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逆的对偶 Brunn-Minkowski 不等式①

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摘要：运用逆的 Hölder 不等式给出了逆的对偶 Minkowski 不等式以及逆的对偶 Brunn-Minkowski 不等式.

关　键　词：逆的 Hölder 不等式；逆的 Minkowski 不等式；逆的对偶 Brunn-Minkowski 不等式

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$\mathbb{R}^n$　$n(n \geqslant 2)$　，B　S^{n-1}　$\mathbb{R}^n$　.　K　$\mathbb{R}^n$　，　$x, y \in K$，　$(1-\lambda)x + \lambda y \in K (0 \leqslant \lambda \leqslant 1)$，　K　.　K　，　K　.　$z \in \operatorname{int} K$，　K　z'　$[z, z']$　K，　K　z　.　z　K

$$\rho(K, z, u) = \max\{\lambda \geqslant 0 : z + \lambda u \in K\} \qquad u \in S^{n-1}$$

K　u　，　K　.　z　o，　K　$\rho(K, o, u)$　$\rho_K(u)$.　$\boldsymbol{S}_o^n$　$\mathbb{R}^n$　，　K　$V(K)$　.　，　.

[1]　，　Brunn-Minkowski　，　Brunn，Minkowski，Blaschke，Aleksandrov　，　、　、Minkowski　、　、　、Minkowski　、　（　[2－5]).　p－　，　，　（　[6－9]).

K_1, K_2　，　$\lambda, \mu \geqslant 0$（　0），　K_1　K_2　Minkowski　$\lambda K_1 + \mu K_2$

$$\lambda K_1 + \mu K_2 = \{\lambda x + \mu y : x \in K_1, y \in K_2\}$$

K　，$\lambda \geqslant 0$，K　Minkowski　$\lambda K = \{\lambda x : x \in K\}$.

$K_1, K_2 \in \boldsymbol{S}_o^n$，　$\lambda, \mu \geqslant 0$（　0），　K_1　K_2　Minkowski　$\lambda K_1 \tilde{+} \mu K_2$　（　[1]）：

$$\lambda K_1 \tilde{+} \mu K_2 = \{\lambda x \tilde{+} \mu y : x \in K_1, y \in K_2\}$$

$$x \tilde{+} y = \begin{cases} x + y & x, o, y \\ o & \end{cases}$$

①　：2015－11－04
：　（　J　LKS[2011]16)；　(11401486，11161007).
：　(1988－)，　，　，　，　.
：　，　.

$$\rho_{\lambda K_1 \widetilde{+} \mu K_2}(u)=\lambda\rho_{K_1}(u)+\mu\rho_{K_2}(u)$$

$K_1, K_2 \in \boldsymbol{S}_o^n$, $\lambda>0$, $K_1=\lambda K_2$, K_1 K_2 .

$K_1, K_2 \in \boldsymbol{S}_o^n$, K_1 K_2 $\widetilde{V}_1(K_1, K_2)$ ([2-3])

$$\widetilde{V}_1(K_1, K_2)=\frac{1}{n}\lim_{\varepsilon\to 0^+}\frac{V(K_1 \widetilde{+} \varepsilon K_2)-V(K_1)}{\varepsilon}=$$

$$\frac{1}{n}\int_{S^{n-1}}\rho_{K_1}(u)^{n-1}\rho_{K_2}(u)\mathrm{d}S(u) \tag{1}$$

[2] Minkowski Brunn-Minkowski , :

$$\widetilde{V}_1(K_1, K_2) \leqslant V(K_1)^{\frac{n-1}{n}}V(K_2)^{\frac{1}{n}} \qquad K_1, K_2 \in \boldsymbol{S}_o^n \tag{2}$$

$$V(K_1 \widetilde{+} K_2)^{\frac{1}{n}} \leqslant V(K_1)^{\frac{1}{n}}+V(K_2)^{\frac{1}{n}} \qquad K_1, K_2 \in \boldsymbol{S}_o^n \tag{3}$$

(2),(3) K_1 K_2 .

$K_1, K_2 \in \boldsymbol{S}_o^n$ $p \in \mathbb{R}\setminus\{0\}$, $\lambda, \mu>0$, K_1 K_2 p- $\lambda \cdot K_1 \widetilde{+}_p \mu \cdot K_2$

$$\rho_{\lambda\cdot K_1 \widetilde{+}_p \mu\cdot K_2}(u)^p=\lambda\rho_{K_1}(u)^p+\mu\rho_{K_2}(u)^p$$

$\lambda \cdot K=\lambda^{\frac{1}{p}}K$. (1) p- $\widetilde{V}_p(K_1, K_2)$ ([7])

$$\widetilde{V}_p(K_1, K_2)=\frac{p}{n}\lim_{\varepsilon\to 0^+}\frac{V(K_1 \widetilde{+}_p \varepsilon\cdot K_2)-V(K_1)}{\varepsilon}=$$

$$\frac{1}{n}\int_{S^{n-1}}\rho_{K_1}(u)^{n-p}\rho_{K_2}(u)^p\mathrm{d}S(u) \tag{4}$$

$p>0$, Hölder [10] (4) p- Minkowski

$$\widetilde{V}_p(K_1, K_2) \leqslant V(K_1)^{\frac{n-p}{n}}V(K_2)^{\frac{p}{n}} \qquad K_1, K_2 \in \boldsymbol{S}_o^n \tag{5}$$

K_1 K_2 .

Minkowski p- Brunn-Minkowski ([7])

$$V(K_1 \widetilde{+}_p K_2)^{\frac{p}{n}} \leqslant V(K_1)^{\frac{p}{n}}+V(K_2)^{\frac{p}{n}} \qquad K_1, K_2 \in \boldsymbol{S}_o^n \tag{6}$$

K_1 K_2 .

(2),(3) (5),(6) , (5) (6) ,

$c_1, c_2 \geqslant 1$, :

$$V(K_1)^{\frac{n-p}{n}}V(K_2)^{\frac{p}{n}} \leqslant c_1\widetilde{V}_p(K_1, K_2) \qquad K_1, K_2 \in \boldsymbol{S}_o^n \tag{7}$$

$$V(K_1)^{\frac{p}{n}}+V(K_2)^{\frac{p}{n}} \leqslant c_2 V(K_1 \widetilde{+}_p K_2)^{\frac{p}{n}} \qquad K_1, K_2 \in \boldsymbol{S}_o^n \tag{8}$$

, [11] , .

([11-12]).

引理 1[13] $0<m_1 \leqslant a \leqslant M_1$, $0<m_2 \leqslant b \leqslant M_2$, $\frac{1}{p_1}+\frac{1}{p_2}=1$, $p_i>1(i=1,2)$,

$$\max\{C_{(p_1, p_2)}(M_1, m_2), C_{(p_1, p_2)}(m_1, M_2)\}a^{\frac{1}{p_1}}b^{\frac{1}{p_2}} \geqslant \frac{a}{p_1}+\frac{b}{p_2}$$

$(a, b)=(m_1, M_2)$ $(a, b)=(M_1, m_2)$,

$$C_{(p_1, p_2)}(x, y)=\frac{\dfrac{x}{p_1}+\dfrac{y}{p_2}}{x^{\frac{1}{p_1}}y^{\frac{1}{p_2}}}$$

Young $\frac{x^{p_1}}{p_1}+\frac{y^{p_2}}{p_2}\geqslant xy(x,y\geqslant 0)$ $C_{(p_1,p_2)}(x,y)\geqslant 1$.

$f(x)$ X ，

$$\| f(x)\|_p=\left(\int_X f(x)^p\mathrm{d}x\right)^{\frac{1}{p}} \qquad p\neq 0$$

引理 2(Hölder) $f_i(x)(i=1,2)$ X $m_i\leqslant f_i(x)\leqslant M_i$

. :

$$T_{(p_1,p_2)}(m_i,M_i,X_i)=\max\left\{C_{(p_1,p_2)}\left(\frac{m_1^{p_1}}{X_1},\frac{M_2^{p_2}}{X_2}\right),C_{(p_1,p_2)}\left(\frac{M_1^{p_1}}{X_1},\frac{m_2^{p_2}}{X_2}\right)\right\}$$

$$X_i=\| f\|_{p_i}^{p_i}$$

$p_i>1$，$\frac{1}{p_1}+\frac{1}{p_2}=1$ $f_i(x)^{p_i}$ ，

$$\prod_{i=1}^{2}\| f_i(x)\|_{p_i}\leqslant T_{(p_1,p_2)}(m_i,M_i,X_i)\left\|\prod_{i=1}^{2}f_i(x)\right\|_1 \tag{9}$$

证 $x_i=\frac{f_i(x)^{p_i}}{X_i}$，$X_i=\| f\|_{p_i}^{p_i}$， $\frac{m_i^{p_i}}{X_i}\leqslant x_i\leqslant\frac{M_i^{p_i}}{X_i}$. 1

$$\max\left\{C_{(p_1,p_2)}\left(\frac{M_1^{p_1}}{X_1},\frac{m_2^{p_2}}{X_2}\right),C_{(p_1,p_2)}\left(\frac{m_1^{p_1}}{X_1},\frac{M_2^{p_2}}{X_2}\right)\right\}\frac{f_1(x)f_2(x)}{X_1^{\frac{1}{p_1}}X_2^{\frac{1}{p_2}}}\geqslant\frac{f_1^{p_1}(x)}{p_1X_1}+\frac{f_2^{p_2}(x)}{p_2X_2}$$

$$\max\left\{C_{(p_1,p_2)}\left(\frac{M_1^{p_1}}{X_1},\frac{m_2^{p_2}}{X_2}\right),C_{(p_1,p_2)}\left(\frac{m_1^{p_1}}{X_1},\frac{M_2^{p_2}}{X_2}\right)\right\}\left\|\prod_{i=1}^{2}f_i(x)\right\|_1\geqslant\prod_{i=1}^{2}\| f_i(x)\|_{p_i}$$

$$\max\left\{C_{(p_1,p_2)}\left(\frac{M_1^{p_1}}{X_1},\frac{m_2^{p_2}}{X_2}\right),C_{(p_1,p_2)}\left(\frac{m_1^{p_1}}{X_1},\frac{M_2^{p_2}}{X_2}\right)\right\}=T_{(p_1,p_2)}(m_i,M_i,X_i)$$

(9).

2， p - Minkowski .

定理 1 $K_1,K_2\in\boldsymbol{S}_o^n$，$0<p<n$，$T_{\left(\frac{n}{n-p},\frac{n}{p}\right)}(m_i,M_i,X_i)$ 2，

$$V(K_1)^{\frac{n-p}{n}}V(K_2)^{\frac{p}{n}}\leqslant T_{\left(\frac{n}{n-p},\frac{n}{p}\right)}(m_i,M_i,X_i)\widetilde{V}_p(K_1,K_2) \tag{10}$$

证 :

$$f_1(u)=\rho_{K_1}(u)^{n-p} \qquad f_2(u)=\rho_{K_2}(u)^p$$

$$p_1=\frac{n}{n-p} \qquad p_2=\frac{n}{p}$$

2 (10).

$p=1$ ， (10) (2) ， :

推论 1 $K_1,K_2\in\boldsymbol{S}_o^n$，$T_{\left(\frac{n}{n-1},n\right)}(m_i,M_i,X_i)$ 2，

$$V(K_1)^{\frac{n-1}{n}}V(K_2)^{\frac{1}{n}}\leqslant T_{\left(\frac{n}{n-1},n\right)}(m_i,M_i,X_i)\widetilde{V}_1(K_1,K_2)$$

2 Minkowski :

引理 3 $f_i(x)(i=1,2)$ X $m_i\leqslant f_i(x)\leqslant M_i$. $p>1$，

:

$$w=\min\left\{\sum_{i=1}^{2}f_i(x)\right\} \qquad W=\max\left\{\sum_{i=1}^{2}f_i(x)\right\} \qquad X_3=\left\|\sum_{i=1}^{2}f_i(x)\right\|_{p-1}^{p-1}$$

$$c_{\left(p,\frac{p}{p-1}\right)}=$$

$$\max\left\{T_{\left(p,\frac{p}{p-1}\right)}(m_1, w^{p-1}, M_1, W^{p-1}, X_1, X_3), T_{\left(p,\frac{p}{p-1}\right)}(m_2, w^{p-1}, M_2, W^{p-1}, X_2, X_3)\right\}$$

$$\sum_{i=1}^{2}\| f_i(x)\|_p \leqslant c_{\left(p,\frac{p}{p-1}\right)}\left\|\sum_{i=1}^{2} f_i(x)\right\|_p \tag{11}$$

证

$$\left\|\sum_{i=1}^{2}f_i(x)\right\|_p^p=\left\|f_1(x)\left(\sum_{i=1}^{2}f_i(x)\right)^{p-1}\right\|_1+\left\|f_2(x)\left(\sum_{i=1}^{2}f_i(x)\right)^{p-1}\right\|_1$$

2, :

$$\|f_1(x)\|_p\left\|\sum_{i=1}^{2}f_i(x)\right\|_p^{p-1}\leqslant$$

$$T_{\left(p,\frac{p}{p-1}\right)}(m_1, w^{p-1}, M_1, W^{p-1}, X_1, X_3)\left\|f_1(x)\left(\sum_{i=1}^{2}f_i(x)\right)^{p-1}\right\|_1 \tag{12}$$

$$\|f_2(x)\|_p\left\|\sum_{i=1}^{2}f_i(x)\right\|_p^{p-1}\leqslant$$

$$T_{\left(p,\frac{p}{p-1}\right)}(m_2, w^{p-1}, M_2, W^{p-1}, X_2, X_3)\left\|f_1(x)\left(\sum_{i=1}^{2}f_i(x)\right)^{p-1}\right\|_1 \tag{13}$$

$$c_{\left(p,\frac{p}{p-1}\right)}=$$

$$\max\left\{T_{\left(p,\frac{p}{p-1}\right)}(m_1, w^{p-1}, M_1, W^{p-1}, X_1, X_3), T_{\left(p,\frac{p}{p-1}\right)}(m_2, w^{p-1}, M_2, W^{p-1}, X_2, X_3)\right\}$$

(12),(13) (11).

3, p - Brunn-Minkowski :

定理 2 $K_1, K_2 \in \boldsymbol{S}_o^n$, $0<p<n$, $c_{\left(\frac{n}{p},\frac{n}{n-p}\right)}$ 3,

$$V(K_1)^{\frac{p}{n}}+V(K_2)^{\frac{p}{n}}\leqslant c_{\left(\frac{n}{p},\frac{n}{n-p}\right)}V(K_1\widetilde{+}_pK_2)^{\frac{p}{n}} \tag{14}$$

证

$$V(K_1\widetilde{+}_pK_2)=\frac{1}{n}\int_{S^{n-1}}[\rho_{K_1}(u)^p+\rho_{K_2}(u)^p]^{\frac{n}{p}}\mathrm{d}x$$

:

$$f_1(u)=\rho_{K_1}(u)^p \qquad f_2(u)=\rho_{K_2}(u)^p$$

3 2.

$p=1$, (14) (3) , :

推论 2 $K_1, K_2 \in \boldsymbol{S}_o^n$, $c_{\left(n,\frac{n}{n-1}\right)}$ 3,

$$V(K_1)^{\frac{1}{n}}+V(K_2)^{\frac{1}{n}}\leqslant c_{\left(n,\frac{n}{n-1}\right)}V(K_1\widetilde{+}K_2)^{\frac{1}{n}}$$

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Reverse Dual Brunn-Minkowski Inequality

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Abstract：In this paper，by applying reverse Hölder's inequality，we obtain the reverse dual Minkowski inequality and the reverse dual Brunn-Minkowski inequality.

Key words：reverse Hölder's inequality；reverse Minkowski inequality；reverse dual Brunn-Minkowski inequality

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