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朝代循环模型的分支方向及稳定性^①

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摘要: 对于具有时滞的朝代循环模型, 当时间延迟到达或穿过临界值时, 系统在正平衡点附近出现了一族周期解. 应用规范型和中心流形理论给出决定该模型分支方向及分支周期解稳定性的显示表达式.

关键词: Hopf 分支; 稳定性; 周期解

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文献[1] 研究了一个具有 3 个种群的社会模型, 模型中 3 个种群分别为农民(x)、强盗(y) 和统治者(z). 文献[2] 在此系统中引入了时间延迟 τ , 证明了时滞系统

$$\begin{cases} \frac{dx}{dt} = x \left[1 - x(t - \tau) - \frac{y}{B + x} - Hz \right] = x\varphi(x, y, z) \\ \frac{dy}{dt} = Qy \left(\frac{Ex}{B + x} - 1 - \frac{z}{D + y} \right) = Qy\psi(x, y, z) \\ \frac{dz}{dt} = R \left(\frac{xy}{B + x} - Gz \right) = R\varphi(x, y, z) \end{cases} \quad (1)$$

当时间延迟 τ 到达或穿过临界值 $\tau^{(j)}$ (其中 $\tau^{(j)}$ 如文献[1] 中所定义) 时, 在正平衡点附近出现了一族周期解. 这里我们假设当 $\tau = \tau^{(j)}$ 时系统(1) 在正平衡点处出现了 Hopf 分支, 下面应用标准形理论和中心流形理论来研究分支周期解的稳定性^[1-7].

令系统(1) 的正平衡点为 x^*, y^*, z^* . 令 $u_1(t) = x(t) - x^*, u_2(t) = y(t) - y^*, u_3(t) = z(t) - z^*$, $x_i(t) = u_i(\tau t), \tau = \tau^{(j)} + \mu (\mu \in \mathbb{R})$. 系统(1) 等价于系统

$$\begin{cases} \frac{du_1(t)}{dt} = (\tau + \mu)(x^* + u_1(t)) \left[\frac{y^*}{B + x^*} - u_1(t - 1) - \frac{y^* + u_2(t)}{B + x^* + u_1(t)} - Hu_3(t) \right] \\ \frac{du_2(t)}{dt} = (\tau + \mu)Q(y^* + u_2(t)) \left[E \frac{x^* + u_1(t)}{B + x^* + u_1(t)} - \frac{Ex^*}{B + x^*} + \frac{z^*}{D + y^*} - \frac{z^* + u_3(t)}{D + y^* + u_2(t)} \right] \\ \frac{du_3(t)}{dt} = (\tau + \mu)R \left[\frac{(x^* + u_1(t))(y^* + u_2(t))}{B + x^* + u_1(t)} - \frac{x^*y^*}{B + x^*} - Gu_3(t) \right] \end{cases} \quad (2)$$

此时系统(2) 具有平衡点 $(0, 0, 0)$.

定理 1 对系统(2) 应用规范型和中心流形理论给出了参数 g_{ij} 的计算公式, 可以计算下列参数的值:

$$\begin{aligned} c_1(0) &= \frac{i}{2\tau^{(j)}\omega_0} \left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right) + \frac{1}{2}g_{21} & \kappa_2 &= -\frac{\text{Re}\{c_1(0)\}}{\text{Re}\{\lambda'(\tau^{(j)})\}} \\ \beta_2 &= 2\text{Re}\{c_1(0)\} & T_2 &= -\frac{\text{Im}\{c_1(0)\} + \kappa_2 \text{Im}\{\lambda'(\tau^{(j)})\}}{\tau^{(j)}\omega_0} \end{aligned}$$

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这几个参数决定了系统当分支值变化到 $\tau^{(j)}$ 时的分支方向及分支周期解的相关性质. κ_2 决定了分支的方向: $\kappa_2 > 0$ 时, 分支是上临界的, 分支周期解在 $\tau > \tau^{(j)}$ 时出现; $\kappa_2 < 0$ 时, 分支是下临界的, 分支周期解在 $\tau < \tau^{(j)}$ 时出现. β_2 决定了分支周期解的稳定性: $\beta_2 < 0$ 时, 中心流形上的周期解是稳定的; 否则是不稳定的. T_2 决定了分支周期解的变化: $T_2 > 0$ 时, 周期解的周期是增加的; 否则是减少的.

证 选取相空间 $C = C([-1, 0], \mathbb{R}^3)$, 对 $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))^T \in C$, 令

$$\begin{aligned} L_\mu \phi = & (\tau^{(j)} + \mu) \begin{pmatrix} \frac{x^* y^*}{(B+x^*)^2} & -\frac{x^*}{B+x^*} & -x^* H \\ \frac{EQBy^*}{(B+x^*)^2} & \frac{Qy^* z^*}{(D+y^*)^2} & \frac{Qy^*}{D+y^*} \\ \frac{RBy^*}{(B+x^*)^2} & \frac{Rx^*}{B+x^*} & -RG \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} + \\ & (\tau^{(j)} + \mu) \begin{pmatrix} -x^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \\ \phi_3(-1) \end{pmatrix} \end{aligned} \quad (3)$$

$F(\mu, \phi) =$

$$(\tau^{(j)} + \mu) \begin{pmatrix} \frac{B\phi_1(0)}{(B+x^*)^2} \left(1 - \frac{\phi_1(0)}{B+x^*}\right) \left(\frac{y^* \phi_1(0)}{B+x^*} - \phi_2(0) - \phi_1(0)\varphi_1(1) - H\phi_1(0)\phi_3(0)\right) \\ \frac{QEB\phi_1(0)}{(B+x^*)^2} \left(\frac{\phi_1(0)}{B+x^*} - 1\right) \left(\frac{y^* \phi_1(0)}{B+x^*} - \phi_2(0)\right) + \frac{QD\phi_2(0)}{(D+y^*)^2} \left(1 - \frac{\phi_2(0)}{D+y^*}\right) \left(\frac{z^* \phi_2(0)}{D+y^*} - \phi_3(0)\right) \\ \frac{RBy^* \phi_1^2(0)}{(B+x^*)^3} \left(\frac{\phi_1(0)}{B+x^*} - 1\right) + \frac{RB\phi_1(0)\phi_2(0)}{(B+x^*)^2} \left(1 - \frac{\phi_1(0)}{B+x^*}\right) \end{pmatrix} \quad (4)$$

则系统(2)等价于

$$\frac{dx(t)}{dt} = L_\mu(x_t) + F(\mu, x_t) \quad (5)$$

其中 $x_t(\theta) = x(t+\theta)$, $\theta \in [-1, 0]$. 由文献[1]知, 当 $\mu = 0$ 时系统(2)的特征方程有一对纯虚根 $\pm i\tau^{(j)}\omega_0$, 且横截条件成立, 系统在零平衡点处出现了 Hopf 分支. 由 Riesz 表示定理, 存在着分量为有界变差函数的三阶矩阵 $\eta(\theta, \mu)$, 使得

$$L_\mu \phi = \int_{-1}^0 d\eta(\theta, 0)\phi(\theta) \quad \phi \in C \quad (6)$$

事实上, 只要取

$$\begin{aligned} \eta(\theta, \mu) = & (\tau^{(j)} + \mu) \begin{pmatrix} \frac{x^* y^*}{(B+x^*)^2} & -\frac{x^*}{B+x^*} & -x^* H \\ \frac{EQBy^*}{(B+x^*)^2} & \frac{Qy^* z^*}{(D+y^*)^2} & \frac{Qy^*}{D+y^*} \\ \frac{RBy^*}{(B+x^*)^2} & \frac{Rx^*}{B+x^*} & -RG \end{pmatrix} \delta(\theta) + \\ & (\tau^{(j)} + \mu) \begin{pmatrix} -x^* & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \delta(\theta+1) \end{aligned}$$

即可. 对 $\phi \in C^1([-1, 0], \mathbb{R}^3)$, 定义

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta} & \theta \in [-1, 0) \\ \int_{-1}^0 d\eta(\mu, s)\phi(s) & \theta = 0 \end{cases} \quad R(\mu)\phi = \begin{cases} 0 & \theta \in [-1, 0) \\ F(\mu, \phi) & \theta = 0 \end{cases}$$

这样方程(5)可写成如下形式

$$\frac{dx(t)}{dt} = \mathbf{A}(\mu)x_t + \mathbf{R}(\mu)x_t \quad (7)$$

其中, 对于任意 $\psi \in C([0, 1], (\mathbb{R}^3)^*)$, 定义

$$\mathbf{A}^* \psi(s) = \begin{cases} -\frac{d\psi(s)}{ds} & s \in (0, 1] \\ \int_{-1}^0 \psi(-t) d\eta(t, 0) & s = 0 \end{cases}$$

和双线性内积

$$\langle \psi(s), \phi(\theta) \rangle = \overline{\psi}(0)\phi(0) - \int_{-1}^0 \int_{\xi=0}^{\theta} \overline{\psi}(\xi - \theta) d\eta(\theta)\phi(\xi) d\xi \quad (8)$$

其中 $\eta(\theta) = \eta(\theta, 0)$, 则 $\mathbf{A}(0)$ 与 \mathbf{A}^* 互为共轭算子. 令 $\mathbf{q}(\theta) = (1, \alpha, \beta)^T e^{i\omega_0 \tau(j)}$ 是 $\mathbf{A}(0)$ 关于 $i\tau^{(j)}\omega_0$ 的特征向量, 即

$$\mathbf{A}(0)\mathbf{q}(\theta) = i\tau^{(j)}\omega_0\mathbf{q}(\theta)$$

计算得

$$\alpha = -\frac{By^*}{(B+x^*)x^*} + \frac{(i\omega_0 + RG)(B+x^*)\beta}{Rx^*}$$

$$\beta = \frac{R(B+x^*)(x^*e^{-i\omega_0\tau} - i\omega_0) + Ry^*}{(B+x^*)(i\omega_0 + RG + x^*HR)}$$

同理, 设 $\mathbf{q}^*(s) = D(1, \alpha^*, \beta^*)e^{i\omega_0 \tau(j)}$ 是 \mathbf{A}^* 关于 $-i\tau^{(j)}\omega_0$ 的特征向量, 计算得

$$\alpha^* = \frac{(D+y^*)[x^*H + (i\omega_0 + RG)]\beta^*}{Qy^*}$$

$$\beta^* = \frac{(B+x^*)^2(-i\omega_0 - x^*e^{i\omega_0\tau}) - x^*y^* - x^*EBH(D+y^*)}{EB(D+y^*)(i\omega_0 + RG) + RBy^*}$$

由双线性内积(8), 可将 \mathbf{q}^* 规范化. 令 \mathbf{x}_t 为(5)式在 $\mu=0$ 时的解, 定义

$$\mathbf{z}(t) = \langle \mathbf{q}^*, \mathbf{x}_t \rangle \quad \mathbf{W}(t, \theta) = \mathbf{x}_t(\theta) - 2\text{Re}\{z(t)\mathbf{q}(\theta)\} \quad (9)$$

在中心流形 C_0 上, 有

$$\mathbf{W}(t, \theta) = \mathbf{W}(z(t), \overline{z(t)}, \theta)$$

其中

$$\mathbf{W}(z, \overline{z}, \theta) = \mathbf{W}_{20}(\theta) \frac{z^2}{2} + \mathbf{W}_{11}(\theta) z \overline{z} + \mathbf{W}_{02}(\theta) \frac{\overline{z}^2}{2} + \mathbf{W}_{30}(\theta) \frac{z^3}{6} + \mathbf{L}$$

$$\mathbf{W}_{20} = (\mathbf{W}_{20}^{(1)}, \mathbf{W}_{20}^{(2)}, \mathbf{W}_{20}^{(3)}) \quad \mathbf{W}_{11} = (\mathbf{W}_{11}^{(1)}, \mathbf{W}_{11}^{(2)}, \mathbf{W}_{11}^{(3)}) \quad \mathbf{W}_{02} = (\mathbf{W}_{02}^{(1)}, \mathbf{W}_{02}^{(2)}, \mathbf{W}_{02}^{(3)})$$

z 和 \overline{z} 是中心流形 C_0 在 \mathbf{q}^* 和 $\overline{\mathbf{q}^*}$ 方向上的局部坐标. 方程(5)的解为 $x_t \in C_0$. 因为 $\mu=0$, 有

$$\frac{dz}{dt} = i\omega_0 \tau^{(j)} z + \langle \mathbf{q}^*(\theta), \mathbf{F}(0, \mathbf{W}(z, \overline{z}, \theta) + 2\text{Re}\{z\mathbf{q}(\theta)\}) \rangle =$$

$$i\omega_0 \tau^{(j)} z + \langle \overline{\mathbf{q}^*}(0) \mathbf{F}_0(z, \overline{z}) \rangle = i\omega_0 \tau^{(j)} z + g(z, \overline{z}) \quad (10)$$

其中

$$g(z, \overline{z}) = \overline{\mathbf{q}^*}(0) \mathbf{F}_0(z, \overline{z}) = g_{20} \frac{z^2}{2} + g_{11} z \overline{z} + g_{02} \frac{\overline{z}^2}{2} + g_{21} \frac{z^2 \overline{z}}{2} + \dots \quad (11)$$

由(9)式得

$$\mathbf{x}_t(\theta) = (x_{1t}(\theta), x_{2t}(\theta), x_{3t}(\theta))^T = \mathbf{W}(t, \theta) + z\mathbf{q}(\theta) + \overline{z}\overline{\mathbf{q}}(\theta)$$

故

$$x_{1t}(0) = z + \overline{z} + \mathbf{W}_{20}^{(1)}(0) \frac{z^2}{2} + \mathbf{W}_{11}^{(1)}(0) z \overline{z} + \mathbf{W}_{02}^{(1)}(0) \frac{\overline{z}^2}{2} + o(|(z, \overline{z})|^3)$$

$$x_{1t}(-1) = z e^{-i\omega_0 \tau(j)} + \overline{z} e^{i\omega_0 \tau(j)} + \mathbf{W}_{20}^{(1)}(-1) \frac{z^2}{2} + \mathbf{W}_{11}^{(1)}(-1) z \overline{z} + \mathbf{W}_{02}^{(1)}(-1) \frac{\overline{z}^2}{2} + o(|(z, \overline{z})|^3)$$

$$x_{2t}(0) = z\alpha + \bar{z}\bar{\alpha} + W_{20}^{(2)}(0) \frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0) \frac{\bar{z}^2}{2} + o(|(z, \bar{z})|^3)$$

$$x_{3t}(0) = z\beta + \bar{z}\bar{\beta} + W_{20}^{(3)}(0) \frac{z^2}{2} + W_{11}^{(3)}(0) z\bar{z} + W_{02}^{(3)}(0) \frac{\bar{z}^2}{2} + o(|(z, \bar{z})|^3)$$

根据(10)式,得

$$g(z, \bar{z}) = \langle \bar{q}^*(0), F_0(z, \bar{z}) \rangle = \bar{D}\tau^{(j)}(1, \bar{\alpha}^*, \bar{\beta}^*)(m_1, m_2, m_3)^T$$

其中

$$m_1 = \frac{B\phi_1(0)}{(B+x^*)^2} \left(1 - \frac{\phi_1(0)}{B+x^*}\right) \left(\frac{y^*\phi_1(0)}{B+x^*} - \phi_2(0)\right) - \phi_1(0)\phi_1(-1) - H\phi_1(0)\phi_3(0)$$

$$m_2 = \frac{QEB\phi_1(0)}{(B+x^*)^2} \left(\frac{\phi_1(0)}{B+x^*} - 1\right) \left(\frac{y^*\phi_1(0)}{B+x^*} - \phi_2(0)\right) + \frac{QD\phi_2(0)}{(D+y^*)^2} \left(1 - \frac{\phi_2(0)}{D+y^*}\right) \left(\frac{z^*\phi_2(0)}{D+y^*} - \phi_3(0)\right)$$

$$m_3 = \frac{RBy^*\phi_1^2(0)}{(B+x^*)^3} \left(\frac{\phi_1(0)}{B+x^*} - 1\right) + \frac{RB\phi_1(0)\phi_2(0)}{(B+x^*)^2} \left(1 - \frac{\phi_1(0)}{B+x^*}\right)$$

将 $x_{1t}(0), x_{1t}(-1), x_{2t}(0), x_{3t}(0)$ 代入 $g(z, \bar{z})$, 与(11)式比较系数,得

$$g_{20} = 2\bar{D}\tau^{(j)} \left[\frac{By^* - B\alpha(B+x^*)}{(B+x^*)^3} (1 - R\bar{\beta}^*) - e^{-i\omega_0\tau} + H\bar{\beta} \right] +$$

$$\bar{\alpha}^* \left(\frac{-EQBy^*}{(B+x^*)^3} + \frac{QDz^*\alpha^2}{(D+y^*)^3} - \frac{QD\alpha\bar{\beta}}{(D+y^*)^2} + \frac{QEB\alpha}{(B+x^*)^2} \right)$$

$$g_{11} = 2\bar{D}\tau^{(j)} \left[\frac{By^* - B\text{Re}\{\alpha\}(B+x^*)}{(B+x^*)^3} (1 - R\bar{\beta}^*) - \frac{e^{-i\omega_0\tau} + e^{i\omega_0\tau}}{2} - \frac{H(\bar{\beta} + \bar{\beta})}{2} \right] +$$

$$\bar{\alpha}^* \left(\frac{-EQBy^*}{(B+x^*)^3} + \frac{QDz^*\alpha\bar{\alpha}}{(D+y^*)^3} - \frac{QD\text{Re}\{\alpha\bar{\beta}\}}{(D+y^*)^2} + \frac{QEB\text{Re}\{\alpha\}}{(B+x^*)^2} \right)$$

$$g_{02} = 2\bar{D}\tau^{(j)} \left[\frac{By^* - B\alpha(B+x^*)}{(B+x^*)^3} (1 - R\bar{\beta}^*) - e^{i\omega_0\tau} - H\bar{\beta} \right] +$$

$$\bar{\alpha}^* \left(\frac{-EQBy^*}{(B+x^*)^3} + \frac{QDz^*\bar{\alpha}^2}{(D+y^*)^3} - \frac{QD\bar{\alpha}\bar{\beta}}{(D+y^*)^2} + \frac{QEB\bar{\alpha}}{(B+x^*)^2} \right)$$

$$g_{21} = W_{11}^{(1)}(0) \left[\frac{2By^* - B\alpha(B+x^*)}{(B+x^*)^3} - e^{-i\omega_0\tau} - H\bar{\beta} + \frac{\bar{\alpha}^*(EQB\alpha(B+x^*) - 2EQBy^*)}{(B+x^*)^3} \right] +$$

$$W_{20}^{(1)}(0) \left[\frac{2By^* - B\bar{\alpha}(B+x^*)}{2(B+x^*)^3} + \frac{e^{i\omega_0\tau} + H\bar{\beta}}{2} + \frac{\bar{\alpha}^*(EQB\bar{\alpha}(B+x^*)) - 2EQBy^*}{2(B+x^*)^3} \right] +$$

$$W_{11}^{(2)}(0) \left[\frac{-B}{2(B+x^*)^2} + \frac{QD\bar{\alpha}^*(2z^*\alpha - \beta)}{(D+y^*)^3} + \frac{\bar{\alpha}^*EBQ + RB\bar{\beta}^*}{(B+x^*)^2} \right] +$$

$$W_{20}^{(2)}(0) \left[\frac{-B}{2(B+x^*)^2} + \frac{QD\bar{\alpha}^*(2z^*\alpha - \beta)}{2(D+y^*)^3} + \frac{\bar{\alpha}^*EBQ + RB\bar{\beta}^*}{2(B+x^*)^2} \right] -$$

$$W_{11}^{(3)}(0) \left[H + \frac{\bar{\alpha}^*DQ\alpha}{(D+y^*)^3} \right] - W_{20}^{(3)}(0) \left[\frac{H}{2} + \frac{\bar{\alpha}^*DQ\bar{\alpha}}{2(D+y^*)^3} \right] - W_{11}^{(1)}(0) - \frac{1}{2}W_{20}^{(1)}(-1) +$$

$$\frac{\bar{\alpha}QEB[3y^* - (2\alpha + \bar{\alpha})(B+x^*)]}{(B+x^*)^4} - \frac{\bar{\beta}^*RB(\bar{\alpha} + 2\alpha)}{(B+x^*)^3}$$

为了确定 g_{21} , 需要计算 $W_{20}(\theta), W_{11}(\theta)$. 由方程(9),(10)得

$$\frac{dW}{dt} = \frac{dx_t}{dt} - \frac{dz}{dt}q - \frac{d\bar{z}}{dt}\bar{q} = \begin{cases} AW - 2\text{Re}\{\bar{q}^*(0)F_0q(\theta)\} & \theta \in [-1, 0) \\ AW - 2\text{Re}\{\bar{q}^*(0)F_0q(0)\} + F_0 & \theta = 0 \end{cases}$$

$$= AW + H(z, \bar{z}, \theta)$$

(12)

其中

$$\mathbf{H}(z, \bar{z}, \theta) = \mathbf{H}_{20}(\theta) \frac{z^2}{2} + \mathbf{H}_{11}(\theta) z \bar{z} + \mathbf{H}_{02} \frac{\bar{z}^2}{2} + \dots \quad (13)$$

注意到, 在接近原点的中心流形 C_0 上, $\mathbf{W} = \mathbf{W}_z \frac{dz}{dt} + \mathbf{W}_{\bar{z}} \frac{d\bar{z}}{dt}$. 由(12), (13) 式得

$$(\mathbf{A} - 2i\tau^{(j)}\omega_0)\mathbf{W}_{20} = -\mathbf{H}_{20}(\theta) \quad \mathbf{A}\mathbf{W}_{11}(\theta) = -\mathbf{H}_{11}(\theta) \quad (14)$$

对于任意的 $\theta \in [-1, 0)$,

$$\mathbf{H}(z, \bar{z}, \theta) = -\bar{\mathbf{q}}^*(0)\mathbf{F}_0\mathbf{q}(\theta) - \mathbf{q}^*(0)\bar{\mathbf{F}}_0\bar{\mathbf{q}}(\theta) = -g\mathbf{q}(\theta) - \bar{g}\bar{\mathbf{q}}(\theta) \quad (15)$$

与(13)式比较系数, 得

$$\mathbf{H}_{20}(\theta) = -g_{20}\mathbf{q}(\theta) - \bar{g}_{02}\bar{\mathbf{q}}(\theta) \quad \mathbf{H}_{11}(\theta) = -g_{11}\mathbf{q}(\theta) - \bar{g}_{11}\bar{\mathbf{q}}(\theta)$$

由(12), (14) 式及矩阵 \mathbf{A} , 得到

$$\frac{d\mathbf{W}_{20}}{dt} = 2i\omega_0\tau^{(j)}\mathbf{W}_{20}(\theta) + g_{20}\mathbf{q}(\theta) + \bar{g}_{02}\bar{\mathbf{q}}(\theta) \quad (16)$$

因 $\mathbf{q}(\theta) = (1, \alpha, \beta)^T e^{i\theta\omega_0\tau^{(j)}}$, 有

$$\mathbf{W}_{20}(\theta) = \frac{i g_{20}}{\tau^{(j)}\omega_0}\mathbf{q}(0)e^{i\theta\omega_0\tau^{(j)}} + \frac{i \bar{g}_{11}}{3\tau^{(j)}\omega_0}\bar{\mathbf{q}}(0)e^{-i\theta\omega_0\tau^{(j)}} + \mathbf{E}_1 e^{2i\theta\omega_0\tau^{(j)}} \quad (17)$$

其中 $\mathbf{E}_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)})^T \in \mathbb{R}^3$ 为常向量, 同理可得

$$\mathbf{W}_{11}(\theta) = \frac{i g_{11}}{-\tau^{(j)}\omega_0}\mathbf{q}(0)e^{i\theta\omega_0\tau^{(j)}} + \frac{i \bar{g}_{11}}{\tau^{(j)}\omega_0}\bar{\mathbf{q}}(0)e^{-i\theta\omega_0\tau^{(j)}} + \mathbf{E}_2 \quad (18)$$

其中 $\mathbf{E}_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)})^T \in \mathbb{R}^3$ 为常向量. 下面计算 $\mathbf{E}_1, \mathbf{E}_2$. 由(14) 式及 \mathbf{A} 的定义, 得

$$\int_{-1}^0 d\boldsymbol{\eta}(\theta)\mathbf{W}_{20} = 2i\tau^{(j)}\omega_0\mathbf{W}_{20}(0) - \mathbf{H}_{20}(0) \quad \int_{-1}^0 d\boldsymbol{\eta}(\theta)\mathbf{W}_{11} = -\mathbf{H}_{11}(0)$$

其中 $\boldsymbol{\eta}(\theta) = \boldsymbol{\eta}(\theta, 0)$, 则

$$\mathbf{H}_{20}(\theta) = -g_{20}\mathbf{q}(\theta) - \bar{g}_{02}\bar{\mathbf{q}}(\theta) + 2\tau^{(j)} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$$\mathbf{H}_{11}(\theta) = -g_{11}\mathbf{q}(\theta) - \bar{g}_{11}\bar{\mathbf{q}}(\theta) + 2\tau^{(j)} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

其中

$$d_1 = \frac{By^* - B\alpha(B+x^*)}{(B+x^*)^3} - e^{-i\omega_0\tau} - H\beta$$

$$d_2 = -\frac{EQBy^*}{(B+x^*)^3} + \frac{QDz^*\alpha^2}{(D+y^*)^3} - \frac{QD\alpha\beta}{(D+y^*)^2} + \frac{QEB\alpha}{(B+x^*)^2}$$

$$d_3 = \frac{RB\alpha}{(B+x^*)^2} - \frac{RBy^*}{(B+x^*)^3}$$

$$c_1 = \frac{By^* - B\text{Re}\{\alpha\}(B+x^*)}{(B+x^*)^3} - \frac{e^{i\omega_0\tau} + e^{-i\omega_0\tau}}{2} - H\text{Re}\{\beta\}$$

$$c_2 = \frac{-EQBy^*}{(B+x^*)^3} + \frac{QDz^*\bar{\alpha}\alpha}{(D+y^*)^3} - \frac{QD\text{Re}[\bar{\alpha}\beta]}{(D+y^*)^2} + \frac{EQB\text{Re}[\alpha]}{(B+x^*)^2}$$

$$c_3 = \frac{RB\text{Re}\{\alpha\}}{(B+x^*)^2} - \frac{RBy^*}{(B+x^*)^3}$$

注意到

$$\left(i\tau^{(j)}\omega_0\mathbf{I} - \int_{-1}^0 e^{i\theta\omega_0\tau^{(j)}} d\boldsymbol{\eta}(\theta)\right)\mathbf{q}(0) = 0 \quad \left(-i\tau^{(j)}\omega_0\mathbf{I} - \int_{-1}^0 e^{-i\theta\omega_0\tau^{(j)}} d\boldsymbol{\eta}(\theta)\right)\bar{\mathbf{q}}(0) = 0$$

得方程

$$\begin{pmatrix} 2i\omega_0 - \frac{x^* y^*}{(B+x^*)^2} - x^* e^{-2i\omega_0} & \frac{x^*}{B+x^*} & x^* H \\ -\frac{EQBy^*}{(B+x^*)^2} & 2i\omega_0 - \frac{Qy^* z^*}{(D+y^*)^2} & -\frac{Qy^*}{D+y^*} \\ -\frac{RBy^*}{(B+x^*)^2} & -\frac{Rx^*}{B+x^*} & 2i\omega_0 + RG \end{pmatrix} \begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \\ E_1^{(3)} \end{pmatrix} = 2 \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{x^* y^*}{(B+x^*)^2} - x^* & \frac{x^*}{B+x^*} & x^* H \\ -\frac{EQBy^*}{(B+x^*)^2} & -\frac{Qy^* z^*}{(D+y^*)^2} & -\frac{Qy^*}{D+y^*} \\ -\frac{RBy^*}{(B+x^*)^2} & -\frac{Rx^*}{B+x^*} & RG \end{pmatrix} \begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \\ E_2^{(3)} \end{pmatrix} = 2 \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

解方程可得 $E_1^{(1)}, E_1^{(2)}, E_1^{(3)}, E_2^{(1)}, E_2^{(2)}, E_2^{(3)}$.

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The Hopf Bifurcation's Direction and Stability of the Dynastic Cycle Model

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Abstract: The positive equilibrium's stability changes when the delay arrives at or crosses a sequence of critical values in the dynamical model of the dynastic cycle with delay. Employing the normal form theory and the center manifold argument in this paper, we give the explicit formulas that determine the direction and other properties of bifurcation periodic solutions.

Key words: Hopf bifurcation; stability; periodic solution

