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# 耦合矩阵方程的双对称最小二乘解及其最佳逼近①

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**摘要**：由于用矩阵分解的方法求解耦合矩阵方程的双对称最小二乘解比较复杂，所以用迭代算法来求解该方程的双对称最小二乘解并证明了算法的收敛性，同时，极小范数解也可通过选取特殊的初始矩阵得到．利用此算法还可得到任意给定矩阵组的最佳逼近双对称解组．

**关　键　词**：矩阵方程；双对称最小二乘解组；极小范数解组；最佳逼近解组

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$m\times n$ 实矩阵和双对称矩阵集合分别用 $\mathbb{R}^{m\times n}$ 和 $\mathbf{BSR}^{m\times n}$ 表示．矩阵 $\boldsymbol{A}$ 与 $\boldsymbol{B}$ 的 Kronecker 积为 $\boldsymbol{A}\otimes\boldsymbol{B}$；实矩阵 $\boldsymbol{A}=(a_{ij})_{n\times m}$ 与 $\boldsymbol{B}=(b_{ij})_{n\times m}$ 的内积为 $\langle\boldsymbol{A},\boldsymbol{B}\rangle=\mathrm{tr}(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A})$，Frobenius 范数 $\|\boldsymbol{A}\|=\sqrt{\langle\boldsymbol{A},\boldsymbol{A}\rangle}$；$\mathrm{vec}(\boldsymbol{A})$ 表示将矩阵 $\boldsymbol{A}$ 按行拉直构成的列向量，单位矩阵 $\boldsymbol{I}_n=(e_1\quad e_2\quad\cdots\quad e_n)$，记 $\boldsymbol{S}_n=(e_n\quad e_{n-1}\quad\cdots\quad e_1)$．

**定义1**　若矩阵 $\boldsymbol{X}\in\mathbb{R}^{n\times n}$ 的元素满足 $x_{ij}=x_{ji}=x_{n+1-j,\,n+1-i}(i,j=1,2,\cdots,n)$，则称 $\boldsymbol{X}$ 为双对称矩阵．

耦合矩阵方程在线性系统理论中有重要的作用．文献[1-2]用矩阵分解的方法讨论矩阵方程的解，但是利用矩阵分解的方法求解多变量矩阵方程的双对称解很困难．文献[3-4]利用迭代算法讨论了多变量矩阵方程的约束解，但在实际应用中，方程的系数往往来源于实验，很难保证解的存在性，因此本文借鉴文献[5]的思想用迭代算法讨论以下问题：

**问题1**　设 $\boldsymbol{A}_i\in\mathbb{R}^{p\times n_i}$，$\boldsymbol{B}_i\in\mathbb{R}^{n_i\times q}(i=1,2,\cdots,l)$，$\boldsymbol{C}\in\mathbb{R}^{p\times q}$，求矩阵组 $[\boldsymbol{X}_1,\cdots,\boldsymbol{X}_l](\boldsymbol{X}_i\in\mathbf{BSR}^{n_i\times n_i},i=1,2,\cdots,l)$，使得

$$\|\boldsymbol{A}_1\boldsymbol{X}_1\boldsymbol{B}_1+\boldsymbol{A}_2\boldsymbol{X}_2\boldsymbol{B}_2+\cdots+\boldsymbol{A}_l\boldsymbol{X}_l\boldsymbol{B}_l-\boldsymbol{C}\|=\min$$

**问题2**　给定 $[\overline{\boldsymbol{X}}_1,\overline{\boldsymbol{X}}_2,\cdots,\overline{\boldsymbol{X}}_l](\boldsymbol{X}_i\in\mathbb{R}^{n_i\times n_i},(i=1,2,\cdots,l))$，设 $\boldsymbol{S}_E$ 表示问题1的解集合，求 $[\hat{\boldsymbol{X}}_1\quad\hat{\boldsymbol{X}}_2\quad\cdots\quad\hat{\boldsymbol{X}}_l]\in\boldsymbol{S}_E$，$\hat{\boldsymbol{X}}_i\in\mathbf{BSR}^{n_i\times n_i}$，使得

$$\begin{aligned}&\|\hat{\boldsymbol{X}}_1-\overline{\boldsymbol{X}}_1\|^2+\|\hat{\boldsymbol{X}}_2-\overline{\boldsymbol{X}}_2\|^2+\cdots+\|\hat{\boldsymbol{X}}_l-\overline{\boldsymbol{X}}_l\|^2=\\&\min_{[\boldsymbol{X}_1,\boldsymbol{X}_2,\cdots,\boldsymbol{X}_l]\in\boldsymbol{S}_E}\{\|\boldsymbol{X}_1-\overline{\boldsymbol{X}}_1\|^2+\|\boldsymbol{X}_2-\overline{\boldsymbol{X}}_2\|^2+\cdots+\|\boldsymbol{X}_l-\overline{\boldsymbol{X}}_l\|^2\}\end{aligned}$$

**定理1**　矩阵方程

$$\boldsymbol{A}_1\boldsymbol{X}_1\boldsymbol{B}_1+\boldsymbol{A}_2\boldsymbol{X}_2\boldsymbol{B}_2+\cdots+\boldsymbol{A}_l\boldsymbol{X}_l\boldsymbol{B}_l=\boldsymbol{C}\tag{1}$$

有双对称解的充要条件是矩阵方程组

$$\begin{cases}\boldsymbol{A}_1\boldsymbol{X}_1\boldsymbol{B}_1+\boldsymbol{A}_2\boldsymbol{X}_2\boldsymbol{B}_2+\cdots+\boldsymbol{A}_l\boldsymbol{X}_l\boldsymbol{B}_l=\boldsymbol{C}\\\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{X}_1\boldsymbol{A}_1^{\mathrm{T}}+\boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{X}_2\boldsymbol{A}_2^{\mathrm{T}}+\cdots+\boldsymbol{B}_l^{\mathrm{T}}\boldsymbol{X}_l\boldsymbol{A}_l^{\mathrm{T}}=\boldsymbol{C}^{\mathrm{T}}\\\boldsymbol{A}_1\boldsymbol{S}_{n1}\boldsymbol{X}_1\boldsymbol{S}_{n1}\boldsymbol{B}_1+\boldsymbol{A}_2\boldsymbol{S}_{n2}\boldsymbol{X}_2\boldsymbol{S}_{n2}\boldsymbol{B}_2+\cdots+\boldsymbol{A}_l\boldsymbol{S}_{nl}\boldsymbol{X}_l\boldsymbol{S}_{nl}\boldsymbol{B}_l=\boldsymbol{C}\\\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{S}_{n1}\boldsymbol{X}_1\boldsymbol{S}_{n1}\boldsymbol{A}_1^{\mathrm{T}}+\boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{S}_{n2}\boldsymbol{X}_2\boldsymbol{S}_{n2}\boldsymbol{A}_2^{\mathrm{T}}+\cdots+\boldsymbol{B}_l^{\mathrm{T}}\boldsymbol{S}_{nl}\boldsymbol{X}_l\boldsymbol{S}_{nl}\boldsymbol{A}_l^{\mathrm{T}}=\boldsymbol{C}^{\mathrm{T}}\end{cases}\tag{2}$$

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相容.

**证** 若矩阵方程(1)有双对称解 $\boldsymbol{H}_i(i=1,2,\cdots,l)$，则

$$\boldsymbol{H}_i=\boldsymbol{H}_i^{\mathrm{T}}=\boldsymbol{S}_{n_i}\boldsymbol{H}_i\boldsymbol{S}_{n_i}$$

$$\boldsymbol{A}_1\boldsymbol{H}_1\boldsymbol{B}_1+\boldsymbol{A}_2\boldsymbol{H}_2\boldsymbol{B}_2+\cdots+\boldsymbol{A}_l\boldsymbol{H}_l\boldsymbol{B}_l=\boldsymbol{C}$$

显然 $\boldsymbol{H}_i$ 就是方程组(2)的解. 反过来，若方程组(2)相容，即存在 $\boldsymbol{G}_i\in\mathbb{R}^{n_i\times n_i}(i=1,2,\cdots,l)$ 满足矩阵方程组(2)，令

$$\boldsymbol{W}_i=\frac{\boldsymbol{G}i+\boldsymbol{G}_i^{\mathrm{T}}+\boldsymbol{S}_{n_i}\boldsymbol{G}_i\boldsymbol{S}_{n_i}+\boldsymbol{S}_{n_i}\boldsymbol{G}_i^{\mathrm{T}}\boldsymbol{S}_{n_i}}{4}$$

则 $\boldsymbol{W}_i\in\mathbf{BSR}^{n_i\times n_i}(i=1,2,\cdots,l)$，且

$$\boldsymbol{A}_1\boldsymbol{W}_1\boldsymbol{B}_1+\boldsymbol{A}_2\boldsymbol{W}_2\boldsymbol{B}_2+\cdots+\boldsymbol{A}_l\boldsymbol{W}_l\boldsymbol{B}_l=\boldsymbol{C}$$

因此 $\boldsymbol{W}_i\in\mathbf{BSR}^{n_i\times n_i}(i=1,2,\cdots,l)$ 是矩阵方程(1)的解，即矩阵方程(1)有双对称解.

## 1 求解问题 1 的迭代算法

**算法 1**

1) 给定矩阵

$$\boldsymbol{A}_i\in\mathbb{R}^{p\times n_i},\ \boldsymbol{B}_i\in\mathbb{R}^{n_i\times q},\ \boldsymbol{C}\in\mathbb{R}^{p\times q},\ \boldsymbol{X}_i^{(0)}\in\mathbf{BSR}^{n_i\times n_i}(i=1,2,\cdots,l)$$

2) 计算

$$\boldsymbol{R}_1^{(0)}=\boldsymbol{C}-\sum_{i=1}^{l}\boldsymbol{A}_i\boldsymbol{X}_i^{(0)}\boldsymbol{B}_i,\ \boldsymbol{R}_2^{(0)}=(\boldsymbol{R}_1^{(0)})^{\mathrm{T}},\ \boldsymbol{R}_3^{(0)}=C-\sum_{i=1}^{l}\boldsymbol{A}_i\boldsymbol{S}_{n_i}\boldsymbol{X}_i^{(0)}\boldsymbol{S}_{n_i}\boldsymbol{B}_i$$

$$\boldsymbol{R}_4^{(0)}=(\boldsymbol{R}_3^{(0)})^{\mathrm{T}},\ \boldsymbol{R}^{(0)}=\mathrm{diag}(\boldsymbol{R}_1^{(0)},\ \boldsymbol{R}_2^{(0)},\ \boldsymbol{R}_3^{(0)},\ \boldsymbol{R}_4^{(0)}),$$

$$\boldsymbol{P}_i^{(0)}=\boldsymbol{S}_i^{(0)}=\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{R}_1^{(0)}\boldsymbol{B}_i^{\mathrm{T}}+\boldsymbol{B}_i\boldsymbol{R}_2^{(0)}\boldsymbol{A}_i+\boldsymbol{S}_{n_i}\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{R}_3^{(0)}\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{S}_{n_i}+\boldsymbol{S}_{n_i}\boldsymbol{B}_i\boldsymbol{R}_4^{(0)}\boldsymbol{A}_i\boldsymbol{S}_{n_i},$$

$$\boldsymbol{S}^{(0)}=\mathrm{diag}(\boldsymbol{S}_1^{(0)}\quad\boldsymbol{S}_2^{(0)}\quad\cdots\quad\boldsymbol{S}_l^{(0)}),\ \gamma_0=\sum_{i=1}^{l}\|\boldsymbol{S}_i^{(0)}\|^2,$$

$$\boldsymbol{Q}_1^{(0)}=\sum_{r=1}^{l}\boldsymbol{A}_r\boldsymbol{P}_r^{(0)}\boldsymbol{B}_r,\ \boldsymbol{Q}_2^{(0)}=[\boldsymbol{Q}_1^{(0)}]^{\mathrm{T}},\ \boldsymbol{Q}_3^{(0)}=\sum_{r=1}^{l}\boldsymbol{A}_r\boldsymbol{S}_{n_r}\boldsymbol{P}_r^{(0)}\boldsymbol{S}_{n_r}\boldsymbol{B}_r,$$

$$\boldsymbol{Q}_4^{(0)}=[\boldsymbol{Q}_3^{(0)}]^{\mathrm{T}},\ \boldsymbol{Q}^{(0)}=\mathrm{diag}(\boldsymbol{Q}_1^{(0)},\ \boldsymbol{Q}_2^{(0)},\ \boldsymbol{Q}_3^{(0)},\ \boldsymbol{Q}_4^{(0)})$$

$$\alpha_0=\gamma_0/(\|\boldsymbol{Q}_1^{(0)}\|^2+\|\boldsymbol{Q}_2^{(0)}\|^2+\|\boldsymbol{Q}_3^{(0)}\|^2+\|\boldsymbol{Q}_4^{(0)}\|^2)k=0$$

3) 如果 $\gamma_k=0$，停止. 否则 $k=k+1$；

4) 计算

$$\boldsymbol{R}_i^{(k)}=\boldsymbol{R}_i^{(k-1)}-\alpha_{k-1}\boldsymbol{Q}_i^{(k-1)}(i=1,2,3,4),\ \boldsymbol{R}^{(k)}=\mathrm{diag}(\boldsymbol{R}_1^{(k)},\ \boldsymbol{R}_2^{(k)},\ \boldsymbol{R}_3^{(k)},\ \boldsymbol{R}_4^{(k)}),$$

$$\boldsymbol{S}_i^{(k)}=\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{R}_1^{(k)}\boldsymbol{B}_i^{\mathrm{T}}+\boldsymbol{B}_i\boldsymbol{R}_2^{(k)}\boldsymbol{A}_i+\boldsymbol{S}_{n_i}\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{R}_3^{(k)}\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{S}_{n_i}+\boldsymbol{S}_{n_i}\boldsymbol{B}_i\boldsymbol{R}_4^{(k)}\boldsymbol{A}_i\boldsymbol{S}_{ni}$$

$$\boldsymbol{S}^{(k)}=\mathrm{diag}(\boldsymbol{S}_1^{(k)},\ \boldsymbol{S}_2^{(k)},\ \cdots,\ \boldsymbol{S}_l^{(k)}),\ \gamma_k=\sum_{i=1}^{l}\|\boldsymbol{S}_i^{(k)}\|^2,\ \beta_k=\frac{\gamma_k}{\gamma_{k-1}},$$

$$\boldsymbol{P}_i^{(k)}=\boldsymbol{S}_i^{(k)}+\beta_{k-1}\boldsymbol{P}_i^{(k-1)},\ \boldsymbol{X}_i^{(k)}=\boldsymbol{X}_i^{(k-1)}+\alpha_{k-1}\boldsymbol{P}_i^{(k-1)},$$

$$\boldsymbol{Q}_1^{(k)}=\sum_{r=1}^{l}\boldsymbol{A}_r\boldsymbol{P}_r^{(k)}\boldsymbol{B}_r,\ \boldsymbol{Q}_2^{(k)}=[\boldsymbol{Q}_1^{(k)}]^{\mathrm{T}},\ \boldsymbol{Q}_3^{(k)}=\sum_{r=1}^{l}\boldsymbol{A}_r\boldsymbol{S}_{n_r}\boldsymbol{P}_r^{(k)}\boldsymbol{S}_{n_r}\boldsymbol{B}_r,$$

$$\boldsymbol{Q}_4^{(k)}=[\boldsymbol{Q}_3^{(k)}]^{\mathrm{T}},\ \boldsymbol{Q}^{(k)}=\mathrm{diag}(\boldsymbol{Q}_1^{(k)},\ \boldsymbol{Q}_2^{(k)},\ \boldsymbol{Q}_3^{(k)},\ \boldsymbol{Q}_4^{(k)}),$$

$$\alpha_k=\frac{\gamma_k}{\|\boldsymbol{Q}_1^{(k)}\|^2+\|\boldsymbol{Q}_2^{(k)}\|^2+\|\boldsymbol{Q}_3^{(k)}\|^2+\|\boldsymbol{Q}_4^{(k)}\|^2}$$

5) 返回 3).

**引理 1** 对算法 1 中的矩阵 $\boldsymbol{S}^{(i)}$，$\boldsymbol{P}_r^{(i)}(r=1,2,\cdots,l)$，若存在正数 $k$，使得 $\boldsymbol{S}^{(i)}\neq\boldsymbol{O}$，$(i=0,1,2,\cdots,k)$，那么

$$\langle\boldsymbol{S}^{(i)},\ \boldsymbol{S}^{(j)}\rangle=0,\ \langle\boldsymbol{Q}^{(i)},\ \boldsymbol{Q}^{(j)}\rangle=0,\ \sum_{r=1}^{l}\langle\boldsymbol{P}_r^{(i)},\ \boldsymbol{S}_r^{(j)}\rangle=0(i\neq j;\ i,j=0,1,2,\cdots,k)$$

**证**　先证明

$$\langle \boldsymbol{S}^{(i)}, \boldsymbol{S}^{(i+1)} \rangle = 0 \qquad \langle \boldsymbol{Q}^{(i)}, \boldsymbol{Q}^{(i+1)} \rangle = 0$$

$$\sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(i)}, \boldsymbol{S}_r^{(i+1)} \rangle = 0 (i = 0,1,2,\cdots,k)$$

用数学归纳法. 当 $i = 0$ 时

$$\begin{aligned}
\langle \boldsymbol{S}^{(0)}, \boldsymbol{S}^{(1)} \rangle = {} & \| \boldsymbol{S}^{(0)} \|^2 - \alpha_0 (\langle \sum_{r=1}^{l} \boldsymbol{A}_r \boldsymbol{P}_r^{(0)} \boldsymbol{B}_r, \boldsymbol{Q}_1^{(0)} \rangle + \langle \sum_{r=1}^{l} \boldsymbol{A}_r \boldsymbol{S}_{n_r} \boldsymbol{P}_r^{(0)} \boldsymbol{S}_{n_r} \boldsymbol{B}_r, \boldsymbol{Q}_3^{(0)} \rangle) - \\
& \alpha_0 (\langle \sum_{r=1}^{l} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{P}_r^{(0)} \boldsymbol{A}_r^{\mathrm{T}}, \boldsymbol{Q}_2^{(0)} \rangle + \langle \sum_{r=1}^{l} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{n_r} \boldsymbol{P}_r^{(0)} \boldsymbol{S}_{n_r} \boldsymbol{A}_r^{\mathrm{T}}, \boldsymbol{Q}_4^{(0)} \rangle) = \\
& \| \boldsymbol{S}^{(0)} \|^2 - \frac{\alpha_0}{\| \boldsymbol{Q}_1^{(0)} \|^2 + \| \boldsymbol{Q}_2^{(0)} \|^2 + \| \boldsymbol{Q}_3^{(0)} \|^2 + \| \boldsymbol{Q}_4^{(0)} \|^2} = 0
\end{aligned}$$

$$\begin{aligned}
\sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(0)}, \boldsymbol{S}_r^{(1)} \rangle = {} & \| \boldsymbol{S}^{(0)} \|^2 - \alpha_0 (\langle \sum_{i=1}^{l} \boldsymbol{A}_r \boldsymbol{P}_r^{(0)} \boldsymbol{B}_r, \boldsymbol{Q}_1^{(0)} \rangle + \langle \sum_{i=1}^{l} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{P}_r^{(0)} \boldsymbol{A}_r^{\mathrm{T}}, \boldsymbol{Q}_2^{(0)} \rangle) - \\
& \alpha_0 (\langle \sum_{i=1}^{l} \boldsymbol{A}_r \boldsymbol{S}_{n_r} \boldsymbol{P}_r^{(0)} \boldsymbol{S}_{n_r} \boldsymbol{B}_r, \boldsymbol{Q}_1^{(0)} \rangle + \langle \sum_{i=1}^{l} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{n_r} \boldsymbol{P}_r^{(0)} \boldsymbol{S}_{n_r} \boldsymbol{A}_r^{\mathrm{T}}, \boldsymbol{Q}_2^{(0)} \rangle) = 0
\end{aligned}$$

$$\begin{aligned}
\langle \boldsymbol{Q}^{(0)}, \boldsymbol{Q}^{(1)} \rangle = {} & \langle \boldsymbol{Q}_1^{(0)}, \boldsymbol{Q}_1^{(1)} \rangle + \langle \boldsymbol{Q}_2^{(0)}, \boldsymbol{Q}_2^{(1)} \rangle + \langle \boldsymbol{Q}_3^{(0)}, \boldsymbol{Q}_3^{(1)} \rangle + \langle \boldsymbol{Q}_4^{(0)}, \boldsymbol{Q}_4^{(1)} \rangle = \\
& \sum_{r=1}^{l} \langle \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(0)} \boldsymbol{B}_r^{\mathrm{T}} + \boldsymbol{B}_r \boldsymbol{Q}_2^{(0)} \boldsymbol{A}_r + \boldsymbol{S}_{n_r} \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_3^{(0)} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{n_r} + \boldsymbol{S}_{n_r} \boldsymbol{B}_r \boldsymbol{Q}_4^{(0)} \boldsymbol{A}_r \boldsymbol{S}_{n_r}, \boldsymbol{P}_r^{(1)} \rangle = \\
& \sum_{r=1}^{l} \frac{1}{\alpha_0} \langle (\boldsymbol{S}_r^{(0)} - \boldsymbol{S}_r^{(1)}), (\boldsymbol{S}_r^{(1)} + \beta_0 \boldsymbol{S}_r^{(0)}) \rangle = -\frac{1}{\alpha_0} \| \boldsymbol{S}^{(1)} \|^2 + \frac{\beta_0}{\alpha_0} \| \boldsymbol{S}^{(1)} \|^2 = 0
\end{aligned}$$

假定对 $i \leqslant t (0 < t < k)$ 时，结论成立，则

$$\begin{aligned}
\langle \boldsymbol{S}^{(t)}, \boldsymbol{S}^{(t+1)} \rangle = {} & \| \boldsymbol{S}^{(t)} \|^2 - \\
& \alpha_t \sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(t)} - \beta_{t-1} \boldsymbol{P}_r^{(t-1)}, \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(t)} \boldsymbol{B}_r^{\mathrm{T}} + \boldsymbol{B}_r \boldsymbol{Q}_2^{(t)} \boldsymbol{A}_r + \boldsymbol{S}_{n_r} \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_3^{(t)} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{n_r} + \boldsymbol{S}_{n_r} \boldsymbol{B}_r \boldsymbol{Q}_4^{(t)} \boldsymbol{A}_r \boldsymbol{S}_{n_r} \rangle = \\
& \| \boldsymbol{S}^{(t)} \|^2 - \alpha_t (\langle \boldsymbol{Q}^{(t)}, \boldsymbol{Q}^{(t)} \rangle - \beta_{k-1} \langle \boldsymbol{Q}^{(t)}, \boldsymbol{Q}^{(t-1)} \rangle) = \| \boldsymbol{S}^{(t)} \|^2 - \alpha_t \| \boldsymbol{Q}^{(t)} \|^2 = 0
\end{aligned}$$

$$\begin{aligned}
\sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(t)}, \boldsymbol{S}_r^{(t+1)} \rangle = {} & \sum_{r=1}^{l} \langle \boldsymbol{S}_r^{(t)} + \beta_{k-1} \boldsymbol{P}_r^{(t)}, \boldsymbol{S}_r^{(t+1)} \rangle = \\
& -\beta_{t-1} \alpha_t (\langle \sum_{r=1}^{l} \boldsymbol{A}_r \boldsymbol{P}_r^{(t-1)} \boldsymbol{B}_r, \boldsymbol{Q}_1^{(t)} \rangle + \langle \sum_{r=1}^{l} \boldsymbol{A}_r \boldsymbol{S}_{n_r} \boldsymbol{P}_r^{(t-1)} \boldsymbol{S}_{n_r} \boldsymbol{B}_r, \boldsymbol{Q}_3^{(t)} \rangle) - \\
& \beta_{t-1} \alpha_t (\langle \sum_{r=1}^{l} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{P}_r^{(t-1)} \boldsymbol{A}_r^{\mathrm{T}}, \boldsymbol{Q}_2^{(t)} \rangle + \langle \sum_{r=1}^{l} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{n_r} \boldsymbol{P}_r^{(t-1)} \boldsymbol{S}_{n_r} \boldsymbol{A}_r^{\mathrm{T}}, \boldsymbol{Q}_2^{(t)} \rangle) = \\
& -\beta_{t-1} \alpha_t \langle \boldsymbol{Q}^{(t-1)}, \boldsymbol{Q}^{(t)} \rangle = 0
\end{aligned}$$

$$\begin{aligned}
\langle \boldsymbol{Q}^{(t)}, \boldsymbol{Q}^{(t+1)} \rangle = {} & \langle \boldsymbol{Q}_1^{(t)}, \boldsymbol{Q}_1^{(t+1)} \rangle + \langle \boldsymbol{Q}_2^{(t)}, \boldsymbol{Q}_2^{(t+1)} \rangle + \langle \boldsymbol{Q}_3^{(t)}, \boldsymbol{Q}_3^{(t+1)} \rangle + \langle \boldsymbol{Q}_4^{(t)}, \boldsymbol{Q}_4^{(t+1)} \rangle = \\
& \sum_{r=1}^{l} \langle \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(t)} \boldsymbol{B}_r^{\mathrm{T}} + \boldsymbol{B}_r \boldsymbol{Q}_2^{(t)} \boldsymbol{A}_r + \boldsymbol{S}_{n_r} \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(t)} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{n_r} + \boldsymbol{S}_{n_r} \boldsymbol{B}_r \boldsymbol{Q}_2^{(t)} \boldsymbol{A}_r \boldsymbol{S}_{n_r}, \boldsymbol{P}_r^{(t+1)} \rangle = \\
& \sum_{r=1}^{l} \frac{1}{\alpha_t} \langle (\boldsymbol{S}_r^{(t)} - \boldsymbol{S}_r^{(t+1)}), (\boldsymbol{S}_r^{(t+1)} + \beta_t \boldsymbol{P}_r^{(t)}) \rangle = \\
& -\frac{1}{\alpha_t} \| \boldsymbol{S}^{(t+1)} \|^2 + \frac{\beta_t}{\alpha_t} \sum_{r=1}^{l} \langle \boldsymbol{S}_r^{(t)}, \boldsymbol{S}_r^{(t)} + \beta_{k-1} \boldsymbol{P}_r^{(t-1)} \rangle = \\
& -\frac{1}{\alpha_t} \| \boldsymbol{S}^{(t+1)} \|^2 + \frac{\beta_t}{\alpha_t} \| \boldsymbol{S}^{(t)} \|^2 = 0
\end{aligned}$$

根据归纳法原理，对 $i = 0,1,2,\cdots,k$，都有

$$\langle \boldsymbol{S}^{(i)}, \boldsymbol{S}^{(i+1)} \rangle = 0, \langle \boldsymbol{Q}^{(i)}, \boldsymbol{Q}^{(i+1)} \rangle = 0, \sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(i)}, \boldsymbol{S}_r^{(i+1)} \rangle = 0$$

第 2 步：假定当 $0 \leqslant i \leqslant k$，$1 < l < k$ 时，

$$\langle \boldsymbol{S}^{(i)}, \boldsymbol{S}^{(i+l)} \rangle = 0,\ \langle \boldsymbol{Q}^{(i)}, \boldsymbol{Q}^{(i+l)} \rangle = 0,\ \sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(i)}, \boldsymbol{S}_r^{(i+l)} \rangle = 0 (i = 0, 1, 2, \cdots, k)$$

则

$$\begin{aligned}
&\langle \boldsymbol{S}^{(i)}, \boldsymbol{S}^{(i+l+1)} \rangle = \\
&-\alpha_{i+l} \sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(i)} - \beta_{i-1} \boldsymbol{P}_r^{(i-1)}, \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_2^{(i+l)} \boldsymbol{B}_r^{\mathrm{T}} + \boldsymbol{B}_r \boldsymbol{Q}_1^{(i+l)} \boldsymbol{A}_r + \boldsymbol{S}_{n_r} \boldsymbol{A}_1^{\mathrm{T}} \boldsymbol{Q}_3^{(i+l)} \boldsymbol{B}_1^{\mathrm{T}} \boldsymbol{S}_{n_r} + \boldsymbol{S}_{n_r} \boldsymbol{B}_1 \boldsymbol{Q}_4^{(i+l)} A \boldsymbol{S}_{n_r} \rangle = \\
&-\alpha_{i+l} (\langle \boldsymbol{Q}^{(i)}, \boldsymbol{Q}^{(i+l)} \rangle - \beta_{i-1} \langle \boldsymbol{Q}^{(i-1)}, \boldsymbol{Q}^{(i+l)} \rangle) = 0
\end{aligned}$$

$$\begin{aligned}
&\sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(i)}, \boldsymbol{S}_r^{(i+l+1)} \rangle = \\
&\sum_{r=1}^{l} \langle \boldsymbol{S}_r^{(i)} + \beta_{i-1} \boldsymbol{P}_r^{(i-1)}, \boldsymbol{S}_r^{(i+l+1)} \rangle = \\
&\sum_{r=1}^{l} \langle \beta_{i-1} \boldsymbol{P}_r^{(i-1)}, -\alpha_{i+l} (\boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(i+l)} \boldsymbol{B}_r^{\mathrm{T}} + \boldsymbol{B}_r \boldsymbol{Q}_2^{(i+l)} \boldsymbol{A}_r + \boldsymbol{S}_{n_r} \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(i+l)} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{n_r} + \boldsymbol{S}_{n_r} \boldsymbol{B}_r \boldsymbol{Q}_2^{(i+l)} \boldsymbol{A}_r \boldsymbol{S}_{n_r})) \rangle = \\
&-\alpha_{i+l} \beta_{i-1} (\langle \sum_{i=1}^{l} \boldsymbol{A}_r \boldsymbol{P}_r^{(i-1)} \boldsymbol{B}_r, \boldsymbol{Q}_1^{(i+l)} \rangle + \langle \sum_{i=1}^{l} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{P}_r^{(i-1)} \boldsymbol{A}_r^{\mathrm{T}}, \boldsymbol{Q}_2^{(i+l)} \rangle) + \\
&-\alpha_{i+l} \beta_{i-1} (\langle \sum_{i=1}^{l} \boldsymbol{A}_r \boldsymbol{S}_{n_r} \boldsymbol{P}_r^{(i-1)} \boldsymbol{S}_{n_r} \boldsymbol{B}_r, \boldsymbol{Q}_3^{(i+l)} \rangle + \langle \sum_{i=1}^{l} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{n_r} \boldsymbol{P}_r^{(i-1)} \boldsymbol{S}_{n_r} \boldsymbol{A}_r^{\mathrm{T}}, \boldsymbol{Q}_4^{(i+l)} \rangle) = 0
\end{aligned}$$

$$\begin{aligned}
\langle \boldsymbol{Q}^{(i)}, \boldsymbol{Q}^{(i+l+1)} \rangle = &\langle \boldsymbol{Q}_1^{(i)}, \sum_{r=1}^{l} \boldsymbol{A}_r \boldsymbol{P}_r^{(i+l+1)} \boldsymbol{B}_r \rangle + \langle \boldsymbol{Q}_3^{(i)}, \sum_{r=1}^{l} \boldsymbol{A}_r \boldsymbol{S}_{n_r} \boldsymbol{P}_r^{(i+l+1)} \boldsymbol{S}_{n_r} \boldsymbol{B}_r \rangle + \\
&\langle \boldsymbol{Q}_2^{(i)}, \sum_{r=1}^{l} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{P}_r^{(i+l+1)} \boldsymbol{A}_r^{\mathrm{T}} \rangle + \langle \boldsymbol{Q}_2^{(i)}, \sum_{r=1}^{l} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{n_r} \boldsymbol{P}_r^{(i+l+1)} \boldsymbol{S}_{n_r} \boldsymbol{A}_r^{\mathrm{T}} \rangle = \\
&\sum_{r=1}^{l} \langle \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(i)} \boldsymbol{B}_r^{\mathrm{T}}, \boldsymbol{P}_r^{(i+l+1)} \rangle + \sum_{r=1}^{l} \langle \boldsymbol{S}_{n_r} \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(i)} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{n_r}, \boldsymbol{P}_r^{(i+l+1)} \rangle + \\
&\sum_{r=1}^{l} \langle \boldsymbol{S}_{n_r} \boldsymbol{B}_r \boldsymbol{Q}_2^{(i)} \boldsymbol{A}_r \boldsymbol{S}_{n_r}, \boldsymbol{P}_r^{(i+l+1)} \rangle + \sum_{r=1}^{l} \langle \boldsymbol{B}_r \boldsymbol{Q}_2^{(i)} \boldsymbol{A}_r, \boldsymbol{P}_r^{(i+l+1)} \rangle = \\
&\sum_{r=1}^{l} \frac{1}{\alpha_i} \langle (\boldsymbol{S}_r^{(i)} - \boldsymbol{S}_r^{(i+1)}), (\boldsymbol{S}_r^{(i+l+1)} + \beta_k \boldsymbol{P}_r^{(i+l)}) \rangle = 0
\end{aligned}$$

由归纳法得

$$\langle \boldsymbol{S}^{(i)}, \boldsymbol{S}^{(j)} \rangle = 0,\ \langle \boldsymbol{Q}^{(i)}, \boldsymbol{Q}^{(j)} \rangle = 0,\ \sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(i)}, \boldsymbol{S}_r^{(j)} \rangle = 0 (i \neq j;\ i, j = 0, 1, 2, \cdots, k)$$

**引理 2** 对任意初始对称矩阵组$(\boldsymbol{X}_1^{(0)}, \boldsymbol{X}_2^{(0)}, \cdots, \boldsymbol{X}_l^{(0)})$，算法 1 中的矩阵 $\boldsymbol{S}_r^{(i)}$，$\boldsymbol{P}_r^{(i)}$ 满足

$$\sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(i)}, \boldsymbol{S}_r^{(i)} \rangle = \| \boldsymbol{S}^{(i)} \|^2 \qquad i = 0, 1, 2, \cdots$$

**证** 采用数学归纳法，当 $i = 0$ 时

$$\sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(0)}, \boldsymbol{S}_r^{(0)} \rangle = \sum_{r=1}^{l} \langle \boldsymbol{S}_r^{(0)}, \boldsymbol{S}_r^{(0)} \rangle = \| \boldsymbol{S}^{(0)} \|^2$$

假定当 $i = k$ 时结论成立，则当 $i = k + 1$ 时，

$$\begin{aligned}
&\sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(k+1)}, \boldsymbol{S}_r^{(k+1)} \rangle = \\
&\sum_{r=1}^{l} \langle \boldsymbol{S}_r^{(k+1)} + \beta_k \boldsymbol{P}_r^{(k)}, \boldsymbol{S}_r^{(k+1)} \rangle = \sum_{r=1}^{l} \langle \boldsymbol{S}_r^{(k+1)}, \boldsymbol{S}_r^{(k+1)} \rangle + \\
&\beta_k \sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(k)}, \boldsymbol{S}_r^{(k)} - \alpha_k (\boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(k)} \boldsymbol{B}_r^{\mathrm{T}} + \boldsymbol{B}_r \boldsymbol{Q}_2^{(k)} \boldsymbol{A}_r + \boldsymbol{S}_{nr} \boldsymbol{B}_r \boldsymbol{Q}_2^{(i)} \boldsymbol{A}_r \boldsymbol{S}_{nr} + \boldsymbol{S}_{nr} \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(k)} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{nr})) \rangle =
\end{aligned}$$

$$\|\boldsymbol{S}^{(k+1)}\|^2+\beta_k\|\boldsymbol{S}^{(k)}\|^2-\beta_k\alpha_k[\sum_{r=1}^{l}\langle\boldsymbol{A}_r\boldsymbol{P}_r^{(k)}\boldsymbol{B}_r,\boldsymbol{Q}_1^{(k)}\rangle+\sum_{r=1}^{l}\langle\boldsymbol{A}_r\boldsymbol{S}_{nr}\boldsymbol{P}_r^{(k)}\boldsymbol{S}_{nr}\boldsymbol{B}_r,\boldsymbol{Q}_3^{(k)}]\rangle-$$

$$\beta_k\alpha_k[\sum_{r=1}^{l}\langle\boldsymbol{B}_r^{\mathrm{T}}\boldsymbol{P}_r^{(k)}\boldsymbol{A}_r^{\mathrm{T}},\boldsymbol{Q}_2^{(k)}\rangle+\sum_{r=1}^{l}\langle\boldsymbol{B}_r^{\mathrm{T}}\boldsymbol{S}_{nr}\boldsymbol{P}_r^{(k)}\boldsymbol{S}_{nr}\boldsymbol{A}_r^{\mathrm{T}},\boldsymbol{Q}_4^{(k)}\rangle]=$$

$$\|\boldsymbol{S}^{(k+1)}\|^2+\beta_k\|\boldsymbol{S}^{(k)}\|^2-\beta_k\gamma_k=\|\boldsymbol{S}^{(k+1)}\|^2$$

由归纳法原理知

$$\sum_{r=1}^{l}\langle\boldsymbol{P}_r^{(i)},\boldsymbol{S}_r^{(i)}\rangle=\|\boldsymbol{S}^{(i)}\|^2\qquad i=0,1,2,\cdots$$

**引理 3**　若$(\boldsymbol{X}_1^*,\boldsymbol{X}_2^*,\cdots,\boldsymbol{X}_l^*)$是耦合矩阵方程的双对称最小二乘解，则对初始矩阵组$(\boldsymbol{X}_1^0,\boldsymbol{X}_2^0,\cdots,\boldsymbol{X}_l^0)(\boldsymbol{X}_i\in\mathbf{BSR}^{n\times n})$，算法中的矩阵 $\boldsymbol{S}_r^{(i)},\boldsymbol{P}_r^{(i)},\boldsymbol{X}_r^{(i)}$ 满足

$$\sum_{r=1}^{l}\langle\boldsymbol{X}_r^*-\boldsymbol{X}_r^{(i)},\boldsymbol{A}_r^{\mathrm{T}}\boldsymbol{Q}_1^{(i)}\boldsymbol{B}_r^{\mathrm{T}}+\boldsymbol{B}_r\boldsymbol{Q}_2^{(i)}\boldsymbol{A}_r+\boldsymbol{S}_{nr}\boldsymbol{A}_r^{\mathrm{T}}\boldsymbol{Q}_3^{(i)}\boldsymbol{B}_r^{\mathrm{T}}\boldsymbol{S}_{nr}+\boldsymbol{S}_{nr}\boldsymbol{B}_r\boldsymbol{Q}_4^{(i)}\boldsymbol{A}_r\boldsymbol{S}_{nr}\rangle=\|\boldsymbol{S}^{(i)}\|^2$$

**证**

$$\sum_{r=1}^{l}\langle\boldsymbol{X}_r^*-\boldsymbol{X}_r^{(i)},\boldsymbol{A}_r^{\mathrm{T}}\boldsymbol{Q}_1^{(i)}\boldsymbol{B}_r^{\mathrm{T}}+\boldsymbol{B}_r\boldsymbol{Q}_2^{(i)}\boldsymbol{A}_r+\boldsymbol{S}_{nr}(\boldsymbol{A}_r^{\mathrm{T}}\boldsymbol{Q}_3^{(i)}\boldsymbol{B}_r^{\mathrm{T}}+\boldsymbol{B}_r\boldsymbol{Q}_4^{(i)}\boldsymbol{A}_r)\boldsymbol{S}_{nr}\rangle=$$

$$\langle\boldsymbol{R}_1^{(i)},\sum_{r=1}^{l}\boldsymbol{A}_r\boldsymbol{P}_r^{(k)}\boldsymbol{B}_r\rangle+\langle\boldsymbol{R}_2^{(i)},\sum_{r=1}^{l}\boldsymbol{B}_r^{\mathrm{T}}\boldsymbol{P}_r^{(k)}\boldsymbol{A}_r^{\mathrm{T}}\rangle+$$

$$\langle\boldsymbol{R}_3^{(i)},\sum_{r=1}^{l}\boldsymbol{A}_r\boldsymbol{S}_{nr}\boldsymbol{P}_r^{(k)}\boldsymbol{S}_{nr}\boldsymbol{B}_r\rangle+\langle\boldsymbol{R}_4^{(i)},\sum_{r=1}^{l}\boldsymbol{B}_r^{\mathrm{T}}\boldsymbol{S}_{nr}\boldsymbol{P}_r^{(k)}\boldsymbol{S}_{nr}\boldsymbol{A}_r^{\mathrm{T}}\rangle=$$

$$\sum_{r=1}^{l}\langle\boldsymbol{S}_r^{(i)},\boldsymbol{P}_r^{(i)}\rangle$$

由引理 2 可得

$$\sum_{r=1}^{l}\langle\boldsymbol{X}_r^*-\boldsymbol{X}_r^{(i)},\boldsymbol{A}_r^{\mathrm{T}}\boldsymbol{Q}_1^{(i)}\boldsymbol{B}_r^{\mathrm{T}}+\boldsymbol{B}_r\boldsymbol{Q}_2^{(i)}\boldsymbol{A}_r+\boldsymbol{S}_{nr}\boldsymbol{A}_r^{\mathrm{T}}\boldsymbol{Q}_3^{(i)}\boldsymbol{B}_r^{\mathrm{T}}\boldsymbol{S}_{nr}+\boldsymbol{S}_{nr}\boldsymbol{B}_r\boldsymbol{Q}_4^{(i)}\boldsymbol{A}_r\boldsymbol{S}_{nr}\rangle=\|\boldsymbol{S}^{(i)}\|^2$$

**注 1**　由引理 3 若 $\boldsymbol{S}^{(i)}\neq O$，

$$\sum_{r=1}^{l}\|\boldsymbol{A}_r^{\mathrm{T}}\boldsymbol{Q}_1^{(i)}\boldsymbol{B}_r^{\mathrm{T}}+\boldsymbol{B}_r\boldsymbol{Q}_2^{(i)}\boldsymbol{A}_r+\boldsymbol{S}_{nr}(\boldsymbol{A}_r^{\mathrm{T}}\boldsymbol{Q}_3^{(i)}\boldsymbol{B}_r^{\mathrm{T}}+\boldsymbol{B}_r\boldsymbol{Q}_4^{(i)}\boldsymbol{A}_r)\boldsymbol{S}_{nr}\|^2\neq 0$$

从而

$$\boldsymbol{Q}^{(i)}\neq O$$

即

$$\sum_{r=1}^{l}\|\boldsymbol{P}_r^{(i)}\|\neq 0$$

**定理 2**　对任意初始矩阵组$(\boldsymbol{X}_1^{(0)},\boldsymbol{X}_2^{(0)},\cdots,\boldsymbol{X}_l^{(0)})(\boldsymbol{X}_i\in\mathbf{BSR}^{n_i\times n_i},i=1,2,\cdots,l)$，通过算法 1 可在有限步迭代后得到问题 1 的解.

**证**　若

$$\boldsymbol{S}^{(i)}\neq\boldsymbol{O}\left(i=0,1,\cdots,t;\ t=\sum_{k=1}^{l}n_k^2\right)$$

由注 1 知

$$\sum_{r=1}^{l}\|\boldsymbol{P}_r^{(i)}\|\neq 0$$

从而可得$(\boldsymbol{X}_1^{(t)},\boldsymbol{X}_2^{(t)},\cdots,\boldsymbol{X}_l^{(t)})$和 $\boldsymbol{S}^{(t)}$. 由引理 1

$$\langle\boldsymbol{S}^{(i)},\boldsymbol{S}^{(t)}\rangle=0\qquad i=1,2,\cdots,t-1$$

而

$$\langle\boldsymbol{S}^{(i)},\boldsymbol{S}^{(j)}\rangle=0\qquad i,j=0,1,\cdots,t-1,\ i\neq j$$

故 $\boldsymbol{S}^{(0)},\boldsymbol{S}^{(1)},\cdots,\boldsymbol{S}^{(t-1)}$ 是空间 $L=\{K \mid K=\operatorname{diag}(K_1, K_2, \cdots, K_l)\}$ 的一组正交基，其中 $K_i \in \mathbb{R}^{n_i \times n_i}$，从而 $\boldsymbol{S}^{(t)}=O$，即$(\boldsymbol{X}_1^t, \boldsymbol{X}_2^t, \cdots, \boldsymbol{X}_l^t)$为矩阵方程(1)的双对称最小二乘解组. 因此问题 1 的解组最多在 $t$ 步计算后得到.

## 2 极小范数解

**引理 4**[5] 设线性方程 $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$ 相容，且有一个解 $\boldsymbol{x}^* \in \mathbb{R}(\boldsymbol{A}^{\mathrm{T}})$，则该方程惟一的极小范数解为 $\boldsymbol{x}^*$.

**定理 3** 若初始矩阵组为

$$\boldsymbol{X}_i^{(0)}=\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{W}_i\boldsymbol{B}_i^{\mathrm{T}}+\boldsymbol{B}_i W_i^{\mathrm{T}}\boldsymbol{A}_i+\boldsymbol{S}_{n_i}(\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{W}_i\boldsymbol{B}_i^{\mathrm{T}}+\boldsymbol{B}_i W_i^{\mathrm{T}}\boldsymbol{A}_i)\boldsymbol{S}_{n_i}\ (i=1,2,\cdots,l)(\boldsymbol{W}_i \in \mathbb{R}^{n_i \times n_i}) \tag{3}$$

则问题 1 的极小范数解由算法 2 在有限步迭代后得到

**证** 选取初始矩阵组为(3)式，则由算法 1 得问题 1 的解组 $\boldsymbol{X}_i^*\ (i=1,2,\cdots,l)$，且 $\boldsymbol{X}_i^*$ 可表示为：

$$\boldsymbol{X}_i^*=\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{Z}_i\boldsymbol{B}_i^{\mathrm{T}}+\boldsymbol{B}_i\boldsymbol{Z}_i^{\mathrm{T}}\boldsymbol{A}_i+\boldsymbol{S}_{n_i}(\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{Z}_i\boldsymbol{B}_i^{\mathrm{T}}+\boldsymbol{B}_i\boldsymbol{Z}_i^{\mathrm{T}}\boldsymbol{A}_i)\boldsymbol{S}_{n_i}\ (i=1,2,\cdots,l)(\boldsymbol{Z}_i \in \mathbb{R}^{n_i \times n_i})$$

因为矩阵方程组(2)等价于线性方程组

$$\begin{pmatrix} \boldsymbol{B}_1^{\mathrm{T}} \otimes \boldsymbol{A}_1 & \boldsymbol{B}_2^{\mathrm{T}} \otimes \boldsymbol{A}_2 & \cdots & \boldsymbol{B}_l^{\mathrm{T}} \otimes \boldsymbol{A}_l \\ \boldsymbol{A}_1 \otimes \boldsymbol{B}_1^{\mathrm{T}} & \boldsymbol{A}_2 \otimes \boldsymbol{B}_2^{\mathrm{T}} & \cdots & \boldsymbol{A}_l \otimes \boldsymbol{B}_l^{\mathrm{T}} \\ \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{S}_{n1} \otimes \boldsymbol{A}_1\boldsymbol{S}_{n1} & \boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{S}_{n2} \otimes \boldsymbol{A}_2\boldsymbol{S}_{n2} & \cdots & \boldsymbol{B}_l^{\mathrm{T}}\boldsymbol{S}_{nl} \otimes \boldsymbol{A}_l\boldsymbol{S}_{nl} \\ \boldsymbol{A}_1\boldsymbol{S}_{n1} \otimes \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{S}_{n1} & \boldsymbol{A}_2\boldsymbol{S}_{n2} \otimes \boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{S}_{n2} & \cdots & \boldsymbol{A}_2\boldsymbol{S}_{nl} \otimes \boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{S}_{nl} \end{pmatrix}$$

$$\begin{pmatrix} \operatorname{vec}(\boldsymbol{X}_1) \\ \operatorname{vec}(\boldsymbol{X}_2) \\ \vdots \\ \operatorname{vec}(\boldsymbol{X}_l) \end{pmatrix}=\begin{pmatrix} \operatorname{vec}(\boldsymbol{C}) \\ \operatorname{vec}(\boldsymbol{C}^{\mathrm{T}}) \\ \operatorname{vec}(\boldsymbol{C}) \\ \operatorname{vec}(\boldsymbol{C}^{\mathrm{T}}) \end{pmatrix} \tag{4}$$

又因为

$$\begin{pmatrix} \operatorname{vec}(\boldsymbol{X}_1^*) \\ \operatorname{vec}(\boldsymbol{X}_2^*) \\ \vdots \\ \operatorname{vec}(\boldsymbol{X}_l^*) \end{pmatrix}=\boldsymbol{M}^{\mathrm{T}}\begin{pmatrix} \operatorname{vec}(\boldsymbol{Z}) \\ \operatorname{vec}(\boldsymbol{Z}^{\mathrm{T}}) \\ \operatorname{vec}(\boldsymbol{Z}) \\ \operatorname{vec}(\boldsymbol{Z}^{\mathrm{T}}) \end{pmatrix} \in \mathbb{R}(\boldsymbol{M}^{\mathrm{T}})$$

其中

$$\boldsymbol{M}=\begin{pmatrix} \boldsymbol{B}_1^{\mathrm{T}} \otimes \boldsymbol{A}_1 & \boldsymbol{B}_2^{\mathrm{T}} \otimes \boldsymbol{A}_2 & \cdots & \boldsymbol{B}_l^{\mathrm{T}} \otimes \boldsymbol{A}_l \\ \boldsymbol{A}_1 \otimes \boldsymbol{B}_1^{\mathrm{T}} & \boldsymbol{A}_2 \otimes \boldsymbol{B}_2^{\mathrm{T}} & \cdots & \boldsymbol{A}_l \otimes \boldsymbol{B}_l^{\mathrm{T}} \\ \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{S}_{n1} \otimes \boldsymbol{A}_1\boldsymbol{S}_{n1} & \boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{S}_{n2} \otimes \boldsymbol{A}_2\boldsymbol{S}_{n2} & \cdots & \boldsymbol{B}_l^{\mathrm{T}}\boldsymbol{S}_{nl} \otimes \boldsymbol{A}_l\boldsymbol{S}_{nl}\boldsymbol{A}_1\boldsymbol{S}_{n1} \otimes \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{S}_{n1} \end{pmatrix}$$

由引理 4 知$(\operatorname{vec}(\boldsymbol{X}_1^*)\quad \operatorname{vec}(\boldsymbol{X}_2^*)\quad \cdots \quad \operatorname{vec}(\boldsymbol{X}_l^*))$是矩阵方程(4)的极小范数解，因此，可得到问题 1 的极小范数解$\lfloor \boldsymbol{X}_1^*, \boldsymbol{X}_2^*, \cdots, \boldsymbol{X}_l^* \rfloor$.

## 3 问题 2 的解

设$[\overline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_2, \cdots, \overline{\boldsymbol{X}}_l](\overline{\boldsymbol{X}}_i \in \mathbb{R}^{n_i \times n_i}\ (i=1,2,\cdots,l))$为任意给定矩阵组，$\boldsymbol{S}_E$ 非空，若 $\boldsymbol{X}_i \in \boldsymbol{S}_E (i=1,2,\cdots,l)$，则问题 2 等价于

$$\| \boldsymbol{X}_1-\overline{\overline{\boldsymbol{X}}}_1 \|^2+\| \boldsymbol{X}_2-\overline{\overline{\boldsymbol{X}}}_2 \|^2+\cdots+\| \boldsymbol{X}_l-\overline{\overline{\boldsymbol{X}}}_l \|^2=\min$$

其中

$$\overline{\overline{\boldsymbol{X}}}_i=\frac{\overline{\boldsymbol{X}}_i+\overline{\boldsymbol{X}}_i^{\mathrm{T}}+\boldsymbol{S}_{n1}(\overline{\boldsymbol{X}}_i+\overline{\boldsymbol{X}}_i^{\mathrm{T}})\boldsymbol{S}_{n1}}{4} \qquad i=1,2,\cdots,l$$

改写矩阵方程组(1)为

$$\begin{cases}\boldsymbol{A}_1\widetilde{\boldsymbol{X}}_1\boldsymbol{B}_1+\boldsymbol{A}_2\widetilde{\boldsymbol{X}}_2\boldsymbol{B}_2+\cdots+\boldsymbol{A}_l\widetilde{\boldsymbol{X}}_l\boldsymbol{B}_l=\widetilde{\boldsymbol{C}}\\ \boldsymbol{B}_1^{\mathrm{T}}\widetilde{\boldsymbol{X}}_1\boldsymbol{A}_1^{\mathrm{T}}+\boldsymbol{B}_2^{\mathrm{T}}\widetilde{\boldsymbol{X}}_2\boldsymbol{A}_2^{\mathrm{T}}+\cdots+\boldsymbol{B}_l^{\mathrm{T}}\widetilde{\boldsymbol{X}}_l\boldsymbol{A}_l^{\mathrm{T}}=\widetilde{\boldsymbol{C}}^{\mathrm{T}}\\ \boldsymbol{A}_1\boldsymbol{S}_{n_1}\widetilde{\boldsymbol{X}}_1\boldsymbol{S}_{n_1}\boldsymbol{B}_1+\boldsymbol{A}_2\boldsymbol{S}_{n_2}\widetilde{\boldsymbol{X}}_2\boldsymbol{S}_{n_2}\boldsymbol{B}_2+\cdots+\boldsymbol{A}_l\boldsymbol{S}_{n_l}\widetilde{\boldsymbol{X}}_l\boldsymbol{S}_{n_l}\boldsymbol{B}_l=\widetilde{\boldsymbol{C}}\\ \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{S}_{n_1}\widetilde{\boldsymbol{X}}_1\boldsymbol{S}_{n_1}\boldsymbol{A}_1^{\mathrm{T}}+\boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{S}_{n_2}\widetilde{\boldsymbol{X}}_2\boldsymbol{S}_{n_2}\boldsymbol{A}_2^{\mathrm{T}}+\cdots+\boldsymbol{B}_l^{\mathrm{T}}\boldsymbol{S}_{n_l}\widetilde{\boldsymbol{X}}_l\boldsymbol{S}_{n_l}\boldsymbol{A}_l^{\mathrm{T}}=\widetilde{\boldsymbol{C}}^{\mathrm{T}}\end{cases}\tag{5}$$

令

$$\widetilde{\boldsymbol{X}}_i=\boldsymbol{X}_i-\overline{\overline{\boldsymbol{X}}}_i,\ \widetilde{\boldsymbol{C}}=\boldsymbol{C}-\sum_{i=1}^{l}\boldsymbol{A}_i\overline{\overline{\boldsymbol{X}}}_i\boldsymbol{B}_i\qquad i=1,2,\cdots,l$$

选初始矩阵(3)可求得矩阵方程组(5)的极小范数解$[\boldsymbol{X}_1^*,\boldsymbol{X}_2^*,\cdots,\boldsymbol{X}_l^*]$，问题 2 的解

$$[\hat{\boldsymbol{X}}_1,\hat{\boldsymbol{X}}_2\cdots\hat{\boldsymbol{X}}_l]=[\widetilde{\boldsymbol{X}}_1^*+\overline{\overline{\boldsymbol{X}}}_1,\widetilde{\boldsymbol{X}}_2^*+\overline{\overline{\boldsymbol{X}}}_2\cdots\widetilde{\boldsymbol{X}}_l^*+\overline{\overline{\boldsymbol{X}}}_l]$$

## 4　数值实验

设 $n=2$，给定 $\boldsymbol{A}_1,\boldsymbol{B}_1,\boldsymbol{A}_2,\boldsymbol{B}_2,\boldsymbol{C}$ 如下.

$$\boldsymbol{A}_1=\begin{bmatrix}3&-5&3&-5&-2&3\\4&3&-1&3&-1&4\\3&-5&3&-5&-2&3\\4&3&-1&3&-1&4\\3&-5&3&-5&2&3\\4&3&-1&3&-1&4\end{bmatrix}$$

$$\boldsymbol{B}_1=\begin{bmatrix}3&9&3&9&3&9\\7&4&7&4&7&4\\4&3&4&3&4&3\\-5&4&-5&4&-5&4\\3&4&3&4&3&4\\4&-5&4&-5&4&-5\end{bmatrix}$$

$$\boldsymbol{A}_2=\begin{bmatrix}4&3&-1&3&-1&4\\3&-5&3&-5&2&5\\4&3&-1&3&-1&4\\3&-5&3&-5&2&5\\4&3&-1&3&-1&4\\3&-5&3&-5&2&5\end{bmatrix}$$

$$\boldsymbol{B}_2=\begin{bmatrix}3&-4&3&-4&3&-4\\5&-3&5&-3&5&-3\\1&4&1&4&1&4\\-5&2&-5&2&-5&2\\-3&4&-3&4&-3&4\\4&-3&4&-3&4&-3\end{bmatrix}$$

$$\boldsymbol{C}=\begin{bmatrix}68&91&54&77&51&83\\81&72&81&72&81&72\\68&91&54&77&51&83\\81&72&81&72&81&72\\68&91&54&77&51&83\\81&72&81&72&81&72\end{bmatrix}$$

$$\overline{\boldsymbol{X}}_1=\begin{bmatrix} -1 & 2 & 1 & -2 & 1 & 5 \\ 2 & 0 & 1 & 0 & -1 & 1 \\ 1 & 4 & 0 & -1 & 5 & 2 \\ 2 & 1 & -3 & 1 & 0 & 1 \\ 3 & -1 & 0 & -1 & 0 & 2 \\ 2 & -1 & -2 & 1 & 2 & -1 \end{bmatrix}$$

$$\overline{\boldsymbol{X}}_2=\begin{bmatrix} -2 & 0 & 3 & -2 & 1 & 1 \\ 0 & -1 & 0 & -2 & 3 & -1 \\ 3 & 2 & -1 & 2 & 0 & 1 \\ 3 & 1 & 2 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 & 5 & 3 \\ 1 & 4 & -2 & 1 & 5 & -2 \end{bmatrix}$$

1）问题 1 的极小范数解：

$$\boldsymbol{X}_1^{(40)}=\begin{bmatrix} 0.731\,9 & 0.505\,8 & 0.376\,1 & 0.066\,4 & 0.463\,2 & 0.731\,9 \\ 0.505\,8 & -0.373\,4 & -0.023\,4 & -0.139\,2 & -0.434\,98 & 0.463\,2 \\ 0.376\,1 & -0.023\,4 & 0.082\,3 & -0.098\,0 & -0.139\,2 & 0.066\,4 \\ 0.066\,4 & -0.139\,2 & -0.098\,0 & 0.082\,3 & -0.023\,4 & 0.376\,1 \\ 0.463\,2 & -0.434\,9 & -0.139\,2 & -0.023\,4 & -0.373\,4 & 0.505\,8 \\ 0.731\,9 & 0.463\,2 & 0.066\,4 & 0.376\,1 & 0.505\,8 & 0.731\,9 \end{bmatrix}$$

$$\overline{\boldsymbol{X}}_2^{(40)}=\begin{bmatrix} -0.047\,6 & 0.113\,5 & 0.031\,3 & 0.151\,7 & 0.093\,0 & -0.086\,9 \\ 0.113\,5 & -0.075\,1 & -0.036\,9 & 0.037\,1 & 0.039\,9 & 0.093\,0 \\ 0.031\,3 & -0.036\,9 & 0.188\,9 & -0.163\,9 & 0.037\,1 & 0.151\,7 \\ 0.151\,7 & 0.037\,1 & -0.163\,9 & 0.188\,9 & -0.075\,1 & 0.113\,5 \\ 0.093\,0 & 0.039\,9 & 0.037\,1 & -0.036\,9 & -0.075\,1 & 0.113\,5 \\ -0.086\,9 & 0.093\,0 & 0.151\,7 & 0.031\,3 & 0.113\,5 & -0.047\,6 \end{bmatrix}$$

$$\|\boldsymbol{R}^{(40)}\|=5.539\times10^{-11},\quad \|\boldsymbol{A}_1\boldsymbol{X}_1^{(40)}\boldsymbol{B}_1+\boldsymbol{A}_2\boldsymbol{X}_2^{(40)}\boldsymbol{B}_2-\boldsymbol{C}\|^2=28.106\,9$$

2）应用算法 1，得到问题 2 的解：

$$\overline{\boldsymbol{X}}_1=\begin{bmatrix} -1.697\,4 & 1.385\,1 & 0.801\,8 & -0.481\,4 & 0.916\,0 & 2.802\,6 \\ 1.385\,1 & 1.013\,6 & 1.089\,5 & 1.530\,5 & -0.349\,2 & 0.916\,0 \\ 0.801\,8 & 1.089\,5 & -0.511\,1 & -1.257\,3 & 1.530\,5 & -0.481\,4 \\ -0.481\,4 & 1.530\,5 & -1.257\,3 & -0.511\,1 & 1.089\,5 & 0.801\,8 \\ 0.916\,0 & -0.349\,2 & 1.530\,5 & 1.089\,5 & 1.013\,6 & 1.385\,1 \\ 2.802\,6 & 0.916\,0 & -0.481\,4 & 0.801\,8 & 1.385\,1 & -1.697\,4 \end{bmatrix}$$

$$\overline{\boldsymbol{X}}_2=\begin{bmatrix} -1.469\,9 & 1.566\,8 & 1.569\,5 & -0.440\,4 & 1.292\,2 & 1.562\,6 \\ 1.566\,8 & 0.968\,3 & 1.044\,4 & -0.296\,3 & 1.959\,9 & 1.292\,2 \\ 1.569\,5 & 1.044\,4 & -1.030\,9 & 1.935\,3 & -0.296\,3 & -0.440\,4 \\ -0.440\,4 & -0.296\,3 & 1.935\,3 & -1.030\,9 & 1.044\,4 & 1.569\,5 \\ 1.292\,2 & 1.959\,9 & -0.296\,3 & 1.044\,4 & 0.968\,3 & 1.566\,8 \\ 1.562\,6 & 1.292\,2 & -0.440\,4 & 1.569\,5 & 1.566\,8 & -1.469\,9 \end{bmatrix}$$

$$\min_{(X,Y)\in \boldsymbol{S}_E}\{\|\boldsymbol{X}_1-\overline{\boldsymbol{X}}_1\|^2+\|\boldsymbol{X}_2-\overline{\boldsymbol{X}}_2\|^2\}=18.428\,0$$

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# The Least Squares Bisymmetric Solution Group of the Coupled Matrix Equation and Its Optimal Approximation

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**Abstract**: Owing to the difficulty of obtaining the bisymmetric solution of a coupled matrix equation by matrix decomposition, an iterative method is proposed in this paper to solve the least squares bisymmetric solution group of the coupled matrix equation, and its convergence is proved. Its least-norm bisymmetric solution group can also be obtained by choosing the special kind of initial iterative matrices. In addition, the unique optimal approximation solution group to the given matrix group in Frobenius norm can be obtained.

**Key words**: matrix equation; least squares bisymmetric solution group; least norm solution group; optimal approximation solution group

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