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耦合矩阵方程的双对称最小二乘解及其最佳逼近①

刘　莉，　王　伟

，　750021

摘要： 由于用矩阵分解的方法求解耦合矩阵方程的双对称最小二乘解比较复杂，所以用迭代算法来求解该方程的双对称最小二乘解并证明了算法的收敛性，同时，极小范数解也可通过选取特殊的初始矩阵得到. 利用此算法还可得到任意给定矩阵组的最佳逼近双对称解组.

关　键　词： 矩阵方程；双对称最小二乘解组；极小范数解组；最佳逼近解组

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$m \times n$ $\mathbb{R}^{m\times n}$ $\mathbf{BSR}^{m\times n}$. $\boldsymbol{A}$ $\boldsymbol{B}$ Kronecker $\boldsymbol{A}\otimes\boldsymbol{B}$；$\boldsymbol{A}=(a_{ij})_{n\times m}$ $\boldsymbol{B}=(b_{ij})_{n\times m}$ $\langle\boldsymbol{A},\boldsymbol{B}\rangle=\mathrm{tr}(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A})$，Frobenius $\|\boldsymbol{A}\|=\sqrt{\langle\boldsymbol{A},\boldsymbol{A}\rangle}$；$\mathrm{vec}(\boldsymbol{A})$ $\boldsymbol{A}$ ， $\boldsymbol{I}_n=(e_1\quad e_2\quad\cdots\quad e_n)$， $\boldsymbol{S}_n=(e_n\quad e_{n-1}\quad\cdots\quad e_1)$.

定义 1 $\boldsymbol{X}\in\mathbb{R}^{n\times n}$ $x_{ij}=x_{ji}=x_{n+1-j,\,n+1-i}(i,j=1,2,\cdots,n)$， $\boldsymbol{X}$ . . [1－2] ， . [3－4]

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[5] ：

问题 1 $\boldsymbol{A}_i\in\mathbb{R}^{p\times n_i}$，$\boldsymbol{B}_i\in\mathbb{R}^{n_i\times q}(i=1,2,\cdots,l)$，$\boldsymbol{C}\in\mathbb{R}^{p\times q}$， $[\boldsymbol{X}_1,\cdots,\boldsymbol{X}_l](\boldsymbol{X}_i\in\mathbf{BSR}^{n_i\times n_i}$，$i=1,2,\cdots,l)$，

$$\|\boldsymbol{A}_1\boldsymbol{X}_1\boldsymbol{B}_1+\boldsymbol{A}_2\boldsymbol{X}_2\boldsymbol{B}_2+\cdots+\boldsymbol{A}_l\boldsymbol{X}_l\boldsymbol{B}_l-\boldsymbol{C}\|=\min$$

问题 2 $[\overline{\boldsymbol{X}}_1,\overline{\boldsymbol{X}}_2,\cdots,\overline{\boldsymbol{X}}_l](\boldsymbol{X}_i\in\mathbb{R}^{n_i\times n_i},(i=1,2,\cdots,l))$， $\boldsymbol{S}_E$ 1 ，$[\hat{\boldsymbol{X}}_1\quad\hat{\boldsymbol{X}}_2\quad\cdots\quad\hat{\boldsymbol{X}}_l]\in\boldsymbol{S}_E$，$\hat{\boldsymbol{X}}_i\in\mathbf{BSR}^{n_i\times n_i}$，

$$\|\hat{\boldsymbol{X}}_1-\overline{\boldsymbol{X}}_1\|^2+\|\hat{\boldsymbol{X}}_2-\overline{\boldsymbol{X}}_2\|^2+\cdots+\|\hat{\boldsymbol{X}}_l-\overline{\boldsymbol{X}}_l\|^2=$$
$$\min_{[\boldsymbol{X}_1,\boldsymbol{X}_2,\cdots,\boldsymbol{X}_l]\in\boldsymbol{S}_E}\{\|\boldsymbol{X}_1-\overline{\boldsymbol{X}}_1\|^2+\|\boldsymbol{X}_2-\overline{\boldsymbol{X}}_2\|^2+\cdots+\|\boldsymbol{X}_l-\overline{\boldsymbol{X}}_l\|^2\}$$

定理 1

$$\boldsymbol{A}_1\boldsymbol{X}_1\boldsymbol{B}_1+\boldsymbol{A}_2\boldsymbol{X}_2\boldsymbol{B}_2+\cdots+\boldsymbol{A}_l\boldsymbol{X}_l\boldsymbol{B}_l=\boldsymbol{C} \tag{1}$$

$$\begin{cases}\boldsymbol{A}_1\boldsymbol{X}_1\boldsymbol{B}_1+\boldsymbol{A}_2\boldsymbol{X}_2\boldsymbol{B}_2+\cdots+\boldsymbol{A}_l\boldsymbol{X}_l\boldsymbol{B}_l=\boldsymbol{C}\\ \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{X}_1\boldsymbol{A}_1^{\mathrm{T}}+\boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{X}_2\boldsymbol{A}_2^{\mathrm{T}}+\cdots+\boldsymbol{B}_l^{\mathrm{T}}\boldsymbol{X}_l\boldsymbol{A}_l^{\mathrm{T}}=\boldsymbol{C}^{\mathrm{T}}\\ \boldsymbol{A}_1\boldsymbol{S}_{n1}\boldsymbol{X}_1\boldsymbol{S}_{n1}\boldsymbol{B}_1+\boldsymbol{A}_2\boldsymbol{S}_{n2}\boldsymbol{X}_2\boldsymbol{S}_{n2}\boldsymbol{B}_2+\cdots+\boldsymbol{A}_l\boldsymbol{S}_{nl}\boldsymbol{X}_l\boldsymbol{S}_{nl}\boldsymbol{B}_l=\boldsymbol{C}\\ \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{S}_{n1}\boldsymbol{X}_1\boldsymbol{S}_{n1}\boldsymbol{A}_1^{\mathrm{T}}+\boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{S}_{n2}\boldsymbol{X}_2\boldsymbol{S}_{n2}\boldsymbol{A}_2^{\mathrm{T}}+\cdots+\boldsymbol{B}_l^{\mathrm{T}}\boldsymbol{S}_{nl}\boldsymbol{X}_l\boldsymbol{S}_{nl}\boldsymbol{A}_l^{\mathrm{T}}=\boldsymbol{C}^{\mathrm{T}}\end{cases} \tag{2}$$

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证 (1) $\boldsymbol{H}_i(i=1,2,\cdots,l)$,

$$\boldsymbol{H}_i=\boldsymbol{H}_i^{\mathrm{T}}=\boldsymbol{S}_{n_i}\boldsymbol{H}_i\boldsymbol{S}_{n_i}$$

$$\boldsymbol{A}_1\boldsymbol{H}_1\boldsymbol{B}_1+\boldsymbol{A}_2\boldsymbol{H}_2\boldsymbol{B}_2+\cdots+\boldsymbol{A}_l\boldsymbol{H}_l\boldsymbol{B}_l=\boldsymbol{C}$$

$\boldsymbol{H}_i$ (2) . , (2) , $\boldsymbol{G}_i\in\mathbb{R}^{n_i\times n_i}(i=1,2,\cdots,l)$ (2),

$$\boldsymbol{W}_i=\frac{\boldsymbol{G}i+\boldsymbol{G}_i^{\mathrm{T}}+\boldsymbol{S}_{n_i}\boldsymbol{G}_i\boldsymbol{S}_{n_i}+\boldsymbol{S}_{n_i}\boldsymbol{G}_i^{\mathrm{T}}\boldsymbol{S}_{n_i}}{4}$$

$\boldsymbol{W}_i\in\mathbf{BSR}^{n_i\times n_i}(i=1,2,\cdots,l)$,

$$\boldsymbol{A}_1\boldsymbol{W}_1\boldsymbol{B}_1+\boldsymbol{A}_2\boldsymbol{W}_2\boldsymbol{B}_2+\cdots+\boldsymbol{A}_l\boldsymbol{W}_l\boldsymbol{B}_l=\boldsymbol{C}$$

$\boldsymbol{W}_i\in\mathbf{BSR}^{n_i\times n_i}(i=1,2,\cdots,l)$ (1) , (1) .

1 求解问题 1 的迭代算法

算法 1

1)

$$\boldsymbol{A}_i\in\mathbb{R}^{p\times n_i},\ \boldsymbol{B}_i\in\mathbb{R}^{n_i\times q},\ \boldsymbol{C}\in\mathbb{R}^{p\times q},\ \boldsymbol{X}_i^{(0)}\in\mathbf{BSR}^{n_i\times n_i}(i=1,2,\cdots,l)$$

2)

$$\boldsymbol{R}_1^{(0)}=\boldsymbol{C}-\sum_{i=1}^{l}\boldsymbol{A}_i\boldsymbol{X}_i^{(0)}\boldsymbol{B}_i,\ \boldsymbol{R}_2^{(0)}=(\boldsymbol{R}_1^{(0)})^{\mathrm{T}},\ \boldsymbol{R}_3^{(0)}=C-\sum_{i=1}^{l}\boldsymbol{A}_i\boldsymbol{S}_{n_i}\boldsymbol{X}_i^{(0)}\boldsymbol{S}_{n_i}\boldsymbol{B}_i$$

$$\boldsymbol{R}_4^{(0)}=(\boldsymbol{R}_3^{(0)})^{\mathrm{T}},\ \boldsymbol{R}^{(0)}=\mathrm{diag}(\boldsymbol{R}_1^{(0)},\ \boldsymbol{R}_2^{(0)},\ \boldsymbol{R}_3^{(0)},\ \boldsymbol{R}_4^{(0)}),$$

$$\boldsymbol{P}_i^{(0)}=\boldsymbol{S}_i^{(0)}=\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{R}_1^{(0)}\boldsymbol{B}_i^{\mathrm{T}}+\boldsymbol{B}_i\boldsymbol{R}_2^{(0)}\boldsymbol{A}_i+\boldsymbol{S}_{n_i}\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{R}_3^{(0)}\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{S}_{n_i}+\boldsymbol{S}_{n_i}\boldsymbol{B}_i\boldsymbol{R}_4^{(0)}\boldsymbol{A}_i\boldsymbol{S}_{n_i},$$

$$\boldsymbol{S}^{(0)}=\mathrm{diag}(\boldsymbol{S}_1^{(0)}\quad \boldsymbol{S}_2^{(0)}\quad \cdots\quad \boldsymbol{S}_l^{(0)}),\ \gamma_0=\sum_{i=1}^{l}\|\boldsymbol{S}_i^{(0)}\|^2,$$

$$\boldsymbol{Q}_1^{(0)}=\sum_{r=1}^{l}\boldsymbol{A}_r\boldsymbol{P}_r^{(0)}\boldsymbol{B}_r,\ \boldsymbol{Q}_2^{(0)}=[\boldsymbol{Q}_1^{(0)}]^{\mathrm{T}},\ \boldsymbol{Q}_3^{(0)}=\sum_{r=1}^{l}\boldsymbol{A}_r\boldsymbol{S}_{n_r}\boldsymbol{P}_r^{(0)}\boldsymbol{S}_{n_r}\boldsymbol{B}_r,$$

$$\boldsymbol{Q}_4^{(0)}=[\boldsymbol{Q}_3^{(0)}]^{\mathrm{T}},\ \boldsymbol{Q}^{(0)}=\mathrm{diag}(\boldsymbol{Q}_1^{(0)},\ \boldsymbol{Q}_2^{(0)},\ \boldsymbol{Q}_3^{(0)},\ \boldsymbol{Q}_4^{(0)})$$

$$\alpha_0=\gamma_0/(\|\boldsymbol{Q}_1^{(0)}\|^2+\|\boldsymbol{Q}_2^{(0)}\|^2+\|\boldsymbol{Q}_3^{(0)}\|^2+\|\boldsymbol{Q}_4^{(0)}\|^2)k=0$$

3) $\gamma_k=0$, . $k=k+1$;

4)

$$\boldsymbol{R}_i^{(k)}=\boldsymbol{R}_i^{(k-1)}-\alpha_{k-1}\boldsymbol{Q}_i^{(k-1)}(i=1,2,3,4),\ \boldsymbol{R}^{(k)}=\mathrm{diag}(\boldsymbol{R}_1^{(k)},\ \boldsymbol{R}_2^{(k)},\ \boldsymbol{R}_3^{(k)},\ \boldsymbol{R}_4^{(k)}),$$

$$\boldsymbol{S}_i^{(k)}=\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{R}_1^{(k)}\boldsymbol{B}_i^{\mathrm{T}}+\boldsymbol{B}_i\boldsymbol{R}_2^{(k)}\boldsymbol{A}_i+\boldsymbol{S}_{n_i}\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{R}_3^{(k)}\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{S}_{n_i}+\boldsymbol{S}_{n_i}\boldsymbol{B}_i\boldsymbol{R}_4^{(k)}\boldsymbol{A}_i\boldsymbol{S}_{ni}$$

$$\boldsymbol{S}^{(k)}=\mathrm{diag}(\boldsymbol{S}_1^{(k)},\ \boldsymbol{S}_2^{(k)},\ \cdots,\ \boldsymbol{S}_l^{(k)}),\ \gamma_k=\sum_{i=1}^{l}\|\boldsymbol{S}_i^{(k)}\|^2,\ \beta_k=\frac{\gamma_k}{\gamma_{k-1}},$$

$$\boldsymbol{P}_i^{(k)}=\boldsymbol{S}_i^{(k)}+\beta_{k-1}\boldsymbol{P}_i^{(k-1)},\ \boldsymbol{X}_i^{(k)}=\boldsymbol{X}_i^{(k-1)}+\alpha_{k-1}\boldsymbol{P}_i^{(k-1)},$$

$$\boldsymbol{Q}_1^{(k)}=\sum_{r=1}^{l}\boldsymbol{A}_r\boldsymbol{P}_r^{(k)}\boldsymbol{B}_r,\ \boldsymbol{Q}_2^{(k)}=[\boldsymbol{Q}_1^{(k)}]^{\mathrm{T}},\ \boldsymbol{Q}_3^{(k)}=\sum_{r=1}^{l}\boldsymbol{A}_r\boldsymbol{S}_{n_r}\boldsymbol{P}_r^{(k)}\boldsymbol{S}_{n_r}\boldsymbol{B}_r,$$

$$\boldsymbol{Q}_4^{(k)}=[\boldsymbol{Q}_3^{(k)}]^{\mathrm{T}},\ \boldsymbol{Q}^{(k)}=\mathrm{diag}(\boldsymbol{Q}_1^{(k)},\ \boldsymbol{Q}_2^{(k)},\ \boldsymbol{Q}_3^{(k)},\ \boldsymbol{Q}_4^{(k)}),$$

$$\alpha_k=\frac{\gamma_k}{\|\boldsymbol{Q}_1^{(k)}\|^2+\|\boldsymbol{Q}_2^{(k)}\|^2+\|\boldsymbol{Q}_3^{(k)}\|^2+\|\boldsymbol{Q}_4^{(k)}\|^2}$$

5) 3).

引理 1 1 $\boldsymbol{S}^{(i)}$, $\boldsymbol{P}_r^{(i)}(r=1,2,\cdots,l)$, k, $\boldsymbol{S}^{(i)}\neq\boldsymbol{O}$, $(i=0,1,2,\cdots,k)$,

$$\langle\boldsymbol{S}^{(i)},\ \boldsymbol{S}^{(j)}\rangle=0,\ \langle\boldsymbol{Q}^{(i)},\ \boldsymbol{Q}^{(j)}\rangle=0,\ \sum_{r=1}^{l}\langle\boldsymbol{P}_r^{(i)},\ \boldsymbol{S}_r^{(j)}\rangle=0(i\neq j;\ i,j=0,1,2,\cdots,k)$$

证

$$\langle \boldsymbol{S}^{(i)}, \boldsymbol{S}^{(i+1)} \rangle = 0 \qquad \langle \boldsymbol{Q}^{(i)}, \boldsymbol{Q}^{(i+1)} \rangle = 0$$

$$\sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(i)}, \boldsymbol{S}_r^{(i+1)} \rangle = 0 (i = 0, 1, 2, \cdots, k)$$

. $i = 0$

$$\begin{aligned}
\langle \boldsymbol{S}^{(0)}, \boldsymbol{S}^{(1)} \rangle = {}& \| \boldsymbol{S}^{(0)} \|^2 - \alpha_0 (\langle \sum_{r=1}^{l} \boldsymbol{A}_r \boldsymbol{P}_r^{(0)} \boldsymbol{B}_r, \boldsymbol{Q}_1^{(0)} \rangle + \langle \sum_{r=1}^{l} \boldsymbol{A}_r \boldsymbol{S}_{n_r} \boldsymbol{P}_r^{(0)} \boldsymbol{S}_{n_r} \boldsymbol{B}_r, \boldsymbol{Q}_3^{(0)} \rangle) - \\
& \alpha_0 (\langle \sum_{r=1}^{l} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{P}_r^{(0)} \boldsymbol{A}_r^{\mathrm{T}}, \boldsymbol{Q}_2^{(0)} \rangle + \langle \sum_{r=1}^{l} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{n_r} \boldsymbol{P}_r^{(0)} \boldsymbol{S}_{n_r} \boldsymbol{A}_r^{\mathrm{T}}, \boldsymbol{Q}_4^{(0)} \rangle) = \\
& \| \boldsymbol{S}^{(0)} \|^2 - \frac{\alpha_0}{\| \boldsymbol{Q}_1^{(0)} \|^2 + \| \boldsymbol{Q}_2^{(0)} \|^2 + \| \boldsymbol{Q}_3^{(0)} \|^2 + \| \boldsymbol{Q}_4^{(0)} \|^2} = 0
\end{aligned}$$

$$\begin{aligned}
\sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(0)}, \boldsymbol{S}_r^{(1)} \rangle = {}& \| \boldsymbol{S}^{(0)} \|^2 - \alpha_0 (\langle \sum_{i=1}^{l} \boldsymbol{A}_r \boldsymbol{P}_r^{(0)} \boldsymbol{B}_r, \boldsymbol{Q}_1^{(0)} \rangle + \langle \sum_{i=1}^{l} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{P}_r^{(0)} \boldsymbol{A}_r^{\mathrm{T}}, \boldsymbol{Q}_2^{(0)} \rangle) - \\
& \alpha_0 (\langle \sum_{i=1}^{l} \boldsymbol{A}_r \boldsymbol{S}_{n_r} \boldsymbol{P}_r^{(0)} \boldsymbol{S}_{n_r} \boldsymbol{B}_r, \boldsymbol{Q}_1^{(0)} \rangle + \langle \sum_{i=1}^{l} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{n_r} \boldsymbol{P}_r^{(0)} \boldsymbol{S}_{n_r} \boldsymbol{A}_r^{\mathrm{T}}, \boldsymbol{Q}_2^{(0)} \rangle) = 0
\end{aligned}$$

$$\begin{aligned}
\langle \boldsymbol{Q}^{(0)}, \boldsymbol{Q}^{(1)} \rangle = {}& \langle \boldsymbol{Q}_1^{(0)}, \boldsymbol{Q}_1^{(1)} \rangle + \langle \boldsymbol{Q}_2^{(0)}, \boldsymbol{Q}_2^{(1)} \rangle + \langle \boldsymbol{Q}_3^{(0)}, \boldsymbol{Q}_3^{(1)} \rangle + \langle \boldsymbol{Q}_4^{(0)}, \boldsymbol{Q}_4^{(1)} \rangle = \\
& \sum_{r=1}^{l} \langle \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(0)} \boldsymbol{B}_r^{\mathrm{T}} + \boldsymbol{B}_r \boldsymbol{Q}_2^{(0)} \boldsymbol{A}_r + \boldsymbol{S}_{n_r} \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_3^{(0)} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{n_r} + \boldsymbol{S}_{n_r} \boldsymbol{B}_r \boldsymbol{Q}_4^{(0)} \boldsymbol{A}_r \boldsymbol{S}_{n_r}, \boldsymbol{P}_r^{(1)} \rangle = \\
& \sum_{r=1}^{l} \frac{1}{\alpha_0} \langle (\boldsymbol{S}_r^{(0)} - \boldsymbol{S}_r^{(1)}), (\boldsymbol{S}_r^{(1)} + \beta_0 \boldsymbol{S}_r^{(0)}) \rangle = -\frac{1}{\alpha_0} \| \boldsymbol{S}^{(1)} \|^2 + \frac{\beta_0}{\alpha_0} \| \boldsymbol{S}^{(1)} \|^2 = 0
\end{aligned}$$

$i \leqslant t (0 < t < k)$ ，　，

$$\begin{aligned}
\langle \boldsymbol{S}^{(t)}, \boldsymbol{S}^{(t+1)} \rangle = {}& \| \boldsymbol{S}^{(t)} \|^2 - \\
& \alpha_t \sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(t)} - \beta_{t-1} \boldsymbol{P}_r^{(t-1)}, \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(t)} \boldsymbol{B}_r^{\mathrm{T}} + \boldsymbol{B}_r \boldsymbol{Q}_2^{(t)} \boldsymbol{A}_r + \boldsymbol{S}_{n_r} \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_3^{(t)} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{n_r} + \boldsymbol{S}_{n_r} \boldsymbol{B}_r \boldsymbol{Q}_4^{(t)} \boldsymbol{A}_r \boldsymbol{S}_{n_r} \rangle = \\
& \| \boldsymbol{S}^{(t)} \|^2 - \alpha_t (\langle \boldsymbol{Q}^{(t)}, \boldsymbol{Q}^{(t)} \rangle - \beta_{k-1} \langle \boldsymbol{Q}^{(t)}, \boldsymbol{Q}^{(t-1)} \rangle) = \| \boldsymbol{S}^{(t)} \|^2 - \alpha_t \| \boldsymbol{Q}^{(t)} \|^2 = 0
\end{aligned}$$

$$\begin{aligned}
\sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(t)}, \boldsymbol{S}_r^{(t+1)} \rangle = {}& \sum_{r=1}^{l} \langle \boldsymbol{S}_r^{(t)} + \beta_{k-1} \boldsymbol{P}_r^{(t)}, \boldsymbol{S}_r^{(t+1)} \rangle = \\
& - \beta_{t-1} \alpha_t (\langle \sum_{r=1}^{l} \boldsymbol{A}_r \boldsymbol{P}_r^{(t-1)} \boldsymbol{B}_r, \boldsymbol{Q}_1^{(t)} \rangle + \langle \sum_{r=1}^{l} \boldsymbol{A}_r \boldsymbol{S}_{n_r} \boldsymbol{P}_r^{(t-1)} \boldsymbol{S}_{n_r} \boldsymbol{B}_r, \boldsymbol{Q}_3^{(t)} \rangle) - \\
& \beta_{t-1} \alpha_t (\langle \sum_{r=1}^{l} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{P}_r^{(t-1)} \boldsymbol{A}_r^{\mathrm{T}}, \boldsymbol{Q}_2^{(t)} \rangle + \langle \sum_{r=1}^{l} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{n_r} \boldsymbol{P}_r^{(t-1)} \boldsymbol{S}_{n_r} \boldsymbol{A}_r^{\mathrm{T}}, \boldsymbol{Q}_2^{(t)} \rangle) = \\
& - \beta_{t-1} \alpha_t \langle \boldsymbol{Q}^{(t-1)}, \boldsymbol{Q}^{(t)} \rangle = 0
\end{aligned}$$

$$\begin{aligned}
\langle \boldsymbol{Q}^{(t)}, \boldsymbol{Q}^{(t+1)} \rangle = {}& \langle \boldsymbol{Q}_1^{(t)}, \boldsymbol{Q}_1^{(t+1)} \rangle + \langle \boldsymbol{Q}_2^{(t)}, \boldsymbol{Q}_2^{(t+1)} \rangle + \langle \boldsymbol{Q}_3^{(t)}, \boldsymbol{Q}_3^{(t+1)} \rangle + \langle \boldsymbol{Q}_4^{(t)}, \boldsymbol{Q}_4^{(t+1)} \rangle = \\
& \sum_{r=1}^{l} \langle \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(t)} \boldsymbol{B}_r^{\mathrm{T}} + \boldsymbol{B}_r \boldsymbol{Q}_2^{(t)} \boldsymbol{A}_r + \boldsymbol{S}_{n_r} \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(t)} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{n_r} + \boldsymbol{S}_{n_r} \boldsymbol{B}_r \boldsymbol{Q}_2^{(t)} \boldsymbol{A}_r \boldsymbol{S}_{n_r}, \boldsymbol{P}_r^{(t+1)} \rangle = \\
& \sum_{r=1}^{l} \frac{1}{\alpha_t} \langle (\boldsymbol{S}_r^{(t)} - \boldsymbol{S}_r^{(t+1)}), (\boldsymbol{S}_r^{(t+1)} + \beta_t \boldsymbol{P}_r^{(t)}) \rangle = \\
& -\frac{1}{\alpha_t} \| \boldsymbol{S}^{(t+1)} \|^2 + \frac{\beta_t}{\alpha_t} \sum_{r=1}^{l} \langle \boldsymbol{S}_r^{(t)}, \boldsymbol{S}_r^{(t)} + \beta_{k-1} \boldsymbol{P}_r^{(t-1)} \rangle = \\
& -\frac{1}{\alpha_t} \| \boldsymbol{S}^{(t+1)} \|^2 + \frac{\beta_t}{\alpha_t} \| \boldsymbol{S}^{(t)} \|^2 = 0
\end{aligned}$$

，　$i = 0, 1, 2, \cdots, k$，

$$\langle \boldsymbol{S}^{(i)}, \boldsymbol{S}^{(i+1)} \rangle = 0, \langle \boldsymbol{Q}^{(i)}, \boldsymbol{Q}^{(i+1)} \rangle = 0, \sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(i)}, \boldsymbol{S}_r^{(i+1)} \rangle = 0$$

2 : $0 \leqslant i \leqslant k,\ 1 < l < k$,

$$\langle \boldsymbol{S}^{(i)}, \boldsymbol{S}^{(i+l)} \rangle = 0,\ \langle \boldsymbol{Q}^{(i)}, \boldsymbol{Q}^{(i+l)} \rangle = 0,\ \sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(i)}, \boldsymbol{S}_r^{(i+l)} \rangle = 0 (i = 0,1,2,\cdots,k)$$

$$\begin{aligned}
&\langle \boldsymbol{S}^{(i)}, \boldsymbol{S}^{(i+l+1)} \rangle = \\
&-\alpha_{i+l} \sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(i)} - \beta_{i-1} \boldsymbol{P}_r^{(i-1)}, \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_2^{(i+l)} \boldsymbol{B}_r^{\mathrm{T}} + \boldsymbol{B}_r \boldsymbol{Q}_1^{(i+l)} \boldsymbol{A}_r + \boldsymbol{S}_{nr} \boldsymbol{A}_1^{\mathrm{T}} \boldsymbol{Q}_3^{(i+l)} \boldsymbol{B}_1^{\mathrm{T}} \boldsymbol{S}_{nr} + \boldsymbol{S}_{nr} \boldsymbol{B}_1 \boldsymbol{Q}_4^{(i+l)} A \boldsymbol{S}_{nr} \rangle = \\
&-\alpha_{i+l} (\langle \boldsymbol{Q}^{(i)}, \boldsymbol{Q}^{(i+l)} \rangle - \beta_{i-1} \langle \boldsymbol{Q}^{(i-1)}, \boldsymbol{Q}^{(i+l)} \rangle) = 0
\end{aligned}$$

$$\begin{aligned}
&\sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(i)}, \boldsymbol{S}_r^{(i+l+1)} \rangle = \\
&\sum_{r=1}^{l} \langle \boldsymbol{S}_r^{(i)} + \beta_{i-1} \boldsymbol{P}_r^{(i-1)}, \boldsymbol{S}_r^{(i+l+1)} \rangle = \\
&\sum_{r=1}^{l} \langle \beta_{i-1} \boldsymbol{P}_r^{(i-1)}, -\alpha_{i+l} (\boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(i+l)} \boldsymbol{B}_r^{\mathrm{T}} + \boldsymbol{B}_r \boldsymbol{Q}_2^{(i+l)} \boldsymbol{A}_r + \boldsymbol{S}_{nr} \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(i+l)} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{nr} + \boldsymbol{S}_{nr} \boldsymbol{B}_r \boldsymbol{Q}_2^{(i+l)} \boldsymbol{A}_r \boldsymbol{S}_{nr}) \rangle = \\
&-\alpha_{i+l} \beta_{i-1} (\langle \sum_{i=1}^{l} \boldsymbol{A}_r \boldsymbol{P}_r^{(i-1)} \boldsymbol{B}_r, \boldsymbol{Q}_1^{(i+l)} \rangle + \langle \sum_{i=1}^{l} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{P}_r^{(i-1)} \boldsymbol{A}_r^{\mathrm{T}}, \boldsymbol{Q}_2^{(i+l)} \rangle) + \\
&-\alpha_{i+l} \beta_{i-1} (\langle \sum_{i=1}^{l} \boldsymbol{A}_r \boldsymbol{S}_{nr} \boldsymbol{P}_r^{(i-1)} \boldsymbol{S}_{nr} \boldsymbol{B}_r, \boldsymbol{Q}_3^{(i+l)} \rangle + \langle \sum_{i=1}^{l} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{nr} \boldsymbol{P}_r^{(i-1)} \boldsymbol{S}_{nr} \boldsymbol{A}_r^{\mathrm{T}}, \boldsymbol{Q}_4^{(i+l)} \rangle) = 0
\end{aligned}$$

$$\begin{aligned}
\langle \boldsymbol{Q}^{(i)}, \boldsymbol{Q}^{(i+l+1)} \rangle = &\langle \boldsymbol{Q}_1^{(i)}, \sum_{r=1}^{l} \boldsymbol{A}_r \boldsymbol{P}_r^{(i+l+1)} \boldsymbol{B}_r \rangle + \langle \boldsymbol{Q}_3^{(i)}, \sum_{r=1}^{l} \boldsymbol{A}_r \boldsymbol{S}_{nr} \boldsymbol{P}_r^{(i+l+1)} \boldsymbol{S}_{nr} \boldsymbol{B}_r \rangle + \\
&\langle \boldsymbol{Q}_2^{(i)}, \sum_{r=1}^{l} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{P}_r^{(i+l+1)} \boldsymbol{A}_r^{\mathrm{T}} \rangle + \langle \boldsymbol{Q}_2^{(i)}, \sum_{r=1}^{l} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{nr} \boldsymbol{P}_r^{(i+l+1)} \boldsymbol{S}_{nr} \boldsymbol{A}_r^{\mathrm{T}} \rangle = \\
&\sum_{r=1}^{l} \langle \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(i)} \boldsymbol{B}_r^{\mathrm{T}}, \boldsymbol{P}_r^{(i+l+1)} \rangle + \sum_{r=1}^{l} \langle \boldsymbol{S}_{nr} \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(i)} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{nr}, \boldsymbol{P}_r^{(i+l+1)} \rangle + \\
&\sum_{r=1}^{l} \langle \boldsymbol{S}_{nr} \boldsymbol{B}_r \boldsymbol{Q}_2^{(i)} \boldsymbol{A}_r \boldsymbol{S}_{nr}, \boldsymbol{P}_r^{(i+l+1)} \rangle + \sum_{r=1}^{l} \langle \boldsymbol{B}_r \boldsymbol{Q}_2^{(i)} \boldsymbol{A}_r, \boldsymbol{P}_r^{(i+l+1)} \rangle = \\
&\sum_{r=1}^{l} \frac{1}{\alpha_i} \langle (\boldsymbol{S}_r^{(i)} - \boldsymbol{S}_r^{(i+1)}), (\boldsymbol{S}_r^{(i+l+1)} + \beta_k \boldsymbol{P}_r^{(i+l)}) \rangle = 0
\end{aligned}$$

$$\langle \boldsymbol{S}^{(i)}, \boldsymbol{S}^{(j)} \rangle = 0,\ \langle \boldsymbol{Q}^{(i)}, \boldsymbol{Q}^{(j)} \rangle = 0,\ \sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(i)}, \boldsymbol{S}_r^{(j)} \rangle = 0 (i \neq j;\ i,j = 0,1,2,\cdots,k)$$

引理 2 $(\boldsymbol{X}_1^{(0)}, \boldsymbol{X}_2^{(0)}, \cdots, \boldsymbol{X}_l^{(0)})$, 1 $\boldsymbol{S}_r^{(i)}, \boldsymbol{P}_r^{(i)}$

$$\sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(i)}, \boldsymbol{S}_r^{(i)} \rangle = \| \boldsymbol{S}^{(i)} \|^2 \qquad i = 0,1,2,\cdots$$

证 , $i = 0$

$$\sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(0)}, \boldsymbol{S}_r^{(0)} \rangle = \sum_{r=1}^{l} \langle \boldsymbol{S}_r^{(0)}, \boldsymbol{S}_r^{(0)} \rangle = \| \boldsymbol{S}^{(0)} \|^2$$

$i = k$, $i = k + 1$,

$$\begin{aligned}
&\sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(k+1)}, \boldsymbol{S}_r^{(k+1)} \rangle = \\
&\sum_{r=1}^{l} \langle \boldsymbol{S}_r^{(k+1)} + \beta_k \boldsymbol{P}_r^{(k)}, \boldsymbol{S}_r^{(k+1)} \rangle = \sum_{r=1}^{l} \langle \boldsymbol{S}_r^{(k+1)}, \boldsymbol{S}_r^{(k+1)} \rangle + \\
&\beta_k \sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(k)}, \boldsymbol{S}_r^{(k)} - \alpha_k (\boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(k)} \boldsymbol{B}_r^{\mathrm{T}} + \boldsymbol{B}_r \boldsymbol{Q}_2^{(k)} \boldsymbol{A}_r + \boldsymbol{S}_{nr} \boldsymbol{B}_r \boldsymbol{Q}_2^{(i)} \boldsymbol{A}_r \boldsymbol{S}_{nr} + \boldsymbol{S}_{nr} \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(k)} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{nr}) \rangle =
\end{aligned}$$

$$\| \boldsymbol{S}^{(k+1)} \|^2 + \beta_k \| \boldsymbol{S}^{(k)} \|^2 - \beta_k \alpha_k [\sum_{r=1}^{l} \langle \boldsymbol{A}_r \boldsymbol{P}_r^{(k)} \boldsymbol{B}_r, \boldsymbol{Q}_1^{(k)} \rangle + \sum_{r=1}^{l} \langle \boldsymbol{A}_r \boldsymbol{S}_{nr} \boldsymbol{P}_r^{(k)} \boldsymbol{S}_{nr} \boldsymbol{B}_r, \boldsymbol{Q}_3^{(k)}]\rangle -$$

$$\beta_k \alpha_k [\sum_{r=1}^{l} \langle \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{P}_r^{(k)} \boldsymbol{A}_r^{\mathrm{T}}, \boldsymbol{Q}_2^{(k)} \rangle + \sum_{r=1}^{l} \langle \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{nr} \boldsymbol{P}_r^{(k)} \boldsymbol{S}_{nr} \boldsymbol{A}_r^{\mathrm{T}}, \boldsymbol{Q}_4^{(k)} \rangle] =$$

$$\| \boldsymbol{S}^{(k+1)} \|^2 + \beta_k \| \boldsymbol{S}^{(k)} \|^2 - \beta_k \gamma_k = \| \boldsymbol{S}^{(k+1)} \|^2$$

$$\sum_{r=1}^{l} \langle \boldsymbol{P}_r^{(i)}, \boldsymbol{S}_r^{(i)} \rangle = \| \boldsymbol{S}^{(i)} \|^2 \qquad i = 0, 1, 2, \cdots$$

引理 3　$(\boldsymbol{X}_1^*, \boldsymbol{X}_2^*, \cdots, \boldsymbol{X}_l^*)$ ，　$(\boldsymbol{X}_1^0, \boldsymbol{X}_2^0, \cdots, \boldsymbol{X}_l^0)(\boldsymbol{X}_i \in \mathbf{BSR}^{n \times n})$，　$\boldsymbol{S}_r^{(i)}, \boldsymbol{P}_r^{(i)}, \boldsymbol{X}_r^{(i)}$

$$\sum_{r=1}^{l} \langle \boldsymbol{X}_r^* - \boldsymbol{X}_r^{(i)}, \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(i)} \boldsymbol{B}_r^{\mathrm{T}} + \boldsymbol{B}_r \boldsymbol{Q}_2^{(i)} \boldsymbol{A}_r + \boldsymbol{S}_{nr} \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_3^{(i)} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{nr} + \boldsymbol{S}_{nr} \boldsymbol{B}_r \boldsymbol{Q}_4^{(i)} \boldsymbol{A}_r \boldsymbol{S}_{nr} \rangle = \| \boldsymbol{S}^{(i)} \|^2$$

证

$$\sum_{r=1}^{l} \langle \boldsymbol{X}_r^* - \boldsymbol{X}_r^{(i)}, \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(i)} \boldsymbol{B}_r^{\mathrm{T}} + \boldsymbol{B}_r \boldsymbol{Q}_2^{(i)} \boldsymbol{A}_r + \boldsymbol{S}_{nr} (\boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_3^{(i)} \boldsymbol{B}_r^{\mathrm{T}} + \boldsymbol{B}_r \boldsymbol{Q}_4^{(i)} \boldsymbol{A}_r) \boldsymbol{S}_{nr} \rangle =$$

$$\langle \boldsymbol{R}_1^{(i)}, \sum_{r=1}^{l} \boldsymbol{A}_r \boldsymbol{P}_r^{(k)} \boldsymbol{B}_r \rangle + \langle \boldsymbol{R}_2^{(i)}, \sum_{r=1}^{l} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{P}_r^{(k)} \boldsymbol{A}_r^{\mathrm{T}} \rangle +$$

$$\langle \boldsymbol{R}_3^{(i)}, \sum_{r=1}^{l} \boldsymbol{A}_r \boldsymbol{S}_{nr} \boldsymbol{P}_r^{(k)} \boldsymbol{S}_{nr} \boldsymbol{B}_r \rangle + \langle \boldsymbol{R}_4^{(i)}, \sum_{r=1}^{l} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{nr} \boldsymbol{P}_r^{(k)} \boldsymbol{S}_{nr} \boldsymbol{A}_r^{\mathrm{T}} \rangle =$$

$$\sum_{r=1}^{l} \langle \boldsymbol{S}_r^{(i)}, \boldsymbol{P}_r^{(i)} \rangle$$

2

$$\sum_{r=1}^{l} \langle \boldsymbol{X}_r^* - \boldsymbol{X}_r^{(i)}, \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(i)} \boldsymbol{B}_r^{\mathrm{T}} + \boldsymbol{B}_r \boldsymbol{Q}_2^{(i)} \boldsymbol{A}_r + \boldsymbol{S}_{nr} \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_3^{(i)} \boldsymbol{B}_r^{\mathrm{T}} \boldsymbol{S}_{nr} + \boldsymbol{S}_{nr} \boldsymbol{B}_r \boldsymbol{Q}_4^{(i)} \boldsymbol{A}_r \boldsymbol{S}_{nr} \rangle = \| \boldsymbol{S}^{(i)} \|^2$$

注 1　　3　$\boldsymbol{S}^{(i)} \neq O$，

$$\sum_{r=1}^{l} \| \boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_1^{(i)} \boldsymbol{B}_r^{\mathrm{T}} + \boldsymbol{B}_r \boldsymbol{Q}_2^{(i)} \boldsymbol{A}_r + \boldsymbol{S}_{nr} (\boldsymbol{A}_r^{\mathrm{T}} \boldsymbol{Q}_3^{(i)} \boldsymbol{B}_r^{\mathrm{T}} + \boldsymbol{B}_r \boldsymbol{Q}_4^{(i)} \boldsymbol{A}_r) \boldsymbol{S}_{nr} \|^2 \neq 0$$

$$\boldsymbol{Q}^{(i)} \neq O$$

$$\sum_{r=1}^{l} \| \boldsymbol{P}_r^{(i)} \| \neq 0$$

定理 2　$(\boldsymbol{X}_1^{(0)}, \boldsymbol{X}_2^{(0)}, \cdots, \boldsymbol{X}_l^{(0)})(\boldsymbol{X}_i \in \mathbf{BSR}^{n_i \times n_i}, i = 1, 2, \cdots, l)$，　1

1　.

证

$$\boldsymbol{S}^{(i)} \neq \boldsymbol{O} \left(i = 0, 1, \cdots, t; t = \sum_{k=1}^{l} n_k^2 \right)$$

1

$$\sum_{r=1}^{l} \| \boldsymbol{P}_r^{(i)} \| \neq 0$$

$(\boldsymbol{X}_1^{(t)}, \boldsymbol{X}_2^{(t)}, \cdots, \boldsymbol{X}_l^{(t)})$　$\boldsymbol{S}^{(t)}$.　1

$$\langle \boldsymbol{S}^{(i)}, \boldsymbol{S}^{(t)} \rangle = 0 \qquad i = 1, 2, \cdots, t-1$$

$$\langle \boldsymbol{S}^{(i)}, \boldsymbol{S}^{(j)} \rangle = 0 \qquad i, j = 0, 1, \cdots, t-1, i \neq j$$

$\boldsymbol{S}^{(0)}, \boldsymbol{S}^{(1)}, \cdots, \boldsymbol{S}^{(t-1)}$ $L=\{K \mid K=\operatorname{diag}(K_1, K_2, \cdots, K_l)\}$, $K_i \in \mathbb{R}^{n_i \times n_i}$, $\boldsymbol{S}^{(t)}=O$, $(\boldsymbol{X}_1^t, \boldsymbol{X}_2^t, \cdots, \boldsymbol{X}_l^t)$ (1) . 1 t .

2 极小范数解

引理 4[5] $\boldsymbol{Ax}=\boldsymbol{b}$, $\boldsymbol{x}^* \in \mathbb{R}(\boldsymbol{A}^{\mathrm{T}})$, $\boldsymbol{x}^*$.

定理 3

$$\boldsymbol{X}_i^{(0)}=\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{W}_i\boldsymbol{B}_i^{\mathrm{T}}+\boldsymbol{B}_i W_i^{\mathrm{T}}\boldsymbol{A}_i+\boldsymbol{S}_{n_i}(\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{W}_i\boldsymbol{B}_i^{\mathrm{T}}+\boldsymbol{B}_i W_i^{\mathrm{T}}\boldsymbol{A}_i)\boldsymbol{S}_{n_i}\ (i=1,2,\cdots,l)(\boldsymbol{W}_i \in \mathbb{R}^{n_i \times n_i}) \tag{3}$$

1 2

证 (3) , 1 1 $\boldsymbol{X}_i^*(i=1,2,\cdots,l)$, $\boldsymbol{X}_i^*$:

$$\boldsymbol{X}_i^*=\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{Z}_i\boldsymbol{B}_i^{\mathrm{T}}+\boldsymbol{B}_i\boldsymbol{Z}_i^{\mathrm{T}}\boldsymbol{A}_i+\boldsymbol{S}_{n_i}(\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{Z}_i\boldsymbol{B}_i^{\mathrm{T}}+\boldsymbol{B}_i\boldsymbol{Z}_i^{\mathrm{T}}\boldsymbol{A}_i)\boldsymbol{S}_{n_i}\ (i=1,2,\cdots,l)(\boldsymbol{Z}_i \in \mathbb{R}^{n_i \times n_i})$$

(2)

$$\begin{pmatrix} \boldsymbol{B}_1^{\mathrm{T}} \otimes \boldsymbol{A}_1 & \boldsymbol{B}_2^{\mathrm{T}} \otimes \boldsymbol{A}_2 & \cdots & \boldsymbol{B}_l^{\mathrm{T}} \otimes \boldsymbol{A}_l \\ \boldsymbol{A}_1 \otimes \boldsymbol{B}_1^{\mathrm{T}} & \boldsymbol{A}_2 \otimes \boldsymbol{B}_2^{\mathrm{T}} & \cdots & \boldsymbol{A}_l \otimes \boldsymbol{B}_l^{\mathrm{T}} \\ \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{S}_{n1} \otimes \boldsymbol{A}_1\boldsymbol{S}_{n1} & \boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{S}_{n2} \otimes \boldsymbol{A}_2\boldsymbol{S}_{n2} & \cdots & \boldsymbol{B}_l^{\mathrm{T}}\boldsymbol{S}_{nl} \otimes \boldsymbol{A}_l\boldsymbol{S}_{nl} \\ \boldsymbol{A}_1\boldsymbol{S}_{n1} \otimes \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{S}_{n1} & \boldsymbol{A}_2\boldsymbol{S}_{n2} \otimes \boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{S}_{n2} & \cdots & \boldsymbol{A}_2\boldsymbol{S}_{nl} \otimes \boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{S}_{nl} \end{pmatrix}$$

$$\begin{pmatrix} \operatorname{vec}(\boldsymbol{X}_1) \\ \operatorname{vec}(\boldsymbol{X}_2) \\ \vdots \\ \operatorname{vec}(\boldsymbol{X}_l) \end{pmatrix} = \begin{pmatrix} \operatorname{vec}(\boldsymbol{C}) \\ \operatorname{vec}(\boldsymbol{C}^{\mathrm{T}}) \\ \operatorname{vec}(\boldsymbol{C}) \\ \operatorname{vec}(\boldsymbol{C}^{\mathrm{T}}) \end{pmatrix} \tag{4}$$

$$\begin{pmatrix} \operatorname{vec}(\boldsymbol{X}_1^*) \\ \operatorname{vec}(\boldsymbol{X}_2^*) \\ \vdots \\ \operatorname{vec}(\boldsymbol{X}_l^*) \end{pmatrix} = \boldsymbol{M}^{\mathrm{T}} \begin{pmatrix} \operatorname{vec}(\boldsymbol{Z}) \\ \operatorname{vec}(\boldsymbol{Z}^{\mathrm{T}}) \\ \operatorname{vec}(\boldsymbol{Z}) \\ \operatorname{vec}(\boldsymbol{Z}^{\mathrm{T}}) \end{pmatrix} \in \mathbb{R}(\boldsymbol{M}^{\mathrm{T}})$$

$$\boldsymbol{M}=\begin{pmatrix} \boldsymbol{B}_1^{\mathrm{T}} \otimes \boldsymbol{A}_1 & \boldsymbol{B}_2^{\mathrm{T}} \otimes \boldsymbol{A}_2 & \cdots & \boldsymbol{B}_l^{\mathrm{T}} \otimes \boldsymbol{A}_l \\ \boldsymbol{A}_1 \otimes \boldsymbol{B}_1^{\mathrm{T}} & \boldsymbol{A}_2 \otimes \boldsymbol{B}_2^{\mathrm{T}} & \cdots & \boldsymbol{A}_l \otimes \boldsymbol{B}_l^{\mathrm{T}} \\ \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{S}_{n1} \otimes \boldsymbol{A}_1\boldsymbol{S}_{n1} & \boldsymbol{B}_2^{\mathrm{T}}\boldsymbol{S}_{n2} \otimes \boldsymbol{A}_2\boldsymbol{S}_{n2} & \cdots & \boldsymbol{B}_l^{\mathrm{T}}\boldsymbol{S}_{nl} \otimes \boldsymbol{A}_l\boldsymbol{S}_{nl}\boldsymbol{A}_1\boldsymbol{S}_{n1} \otimes \boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{S}_{n1} \end{pmatrix}$$

4 $(\operatorname{vec}(\boldsymbol{X}_1^*)\quad \operatorname{vec}(\boldsymbol{X}_2^*)\quad \cdots \quad \operatorname{vec}(\boldsymbol{X}_l^*))$ (4) , , 1

$\lfloor \boldsymbol{X}_1^*, \boldsymbol{X}_2^*, \cdots, \boldsymbol{X}_l^* \rfloor$.

3 问题 2 的解

$[\overline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_2, \cdots, \overline{\boldsymbol{X}}_l](\overline{\boldsymbol{X}}_i \in \mathbb{R}^{n_i \times n_i}(i=1,2,\cdots,l))$, $\boldsymbol{S}_E$, $\boldsymbol{X}_i \in \boldsymbol{S}_E(i=1,2,\cdots,l)$, 2

$$\|\boldsymbol{X}_1-\overline{\overline{\boldsymbol{X}}}_1\|^2+\|\boldsymbol{X}_2-\overline{\overline{\boldsymbol{X}}}_2\|^2+\cdots+\|\boldsymbol{X}_l-\overline{\overline{\boldsymbol{X}}}_l\|^2=\min$$

$$\overline{\overline{\boldsymbol{X}}}_i=\frac{\overline{\boldsymbol{X}}_i+\overline{\boldsymbol{X}}_i^{\mathrm{T}}+\boldsymbol{S}_{n1}(\overline{\boldsymbol{X}}_i+\overline{\boldsymbol{X}}_i^{\mathrm{T}})\boldsymbol{S}_{n1}}{4} \qquad i=1,2,\cdots,l$$

(1)

$$\begin{cases}A_1\widetilde{X}_1B_1+A_2\widetilde{X}_2B_2+\cdots+A_l\widetilde{X}_lB_l=\widetilde{C}\\ B_1^{\mathrm{T}}\widetilde{X}_1A_1^{\mathrm{T}}+B_2^{\mathrm{T}}\widetilde{X}_2A_2^{\mathrm{T}}+\cdots+B_l^{\mathrm{T}}\widetilde{X}_lA_l^{\mathrm{T}}=\widetilde{C}^{\mathrm{T}}\\ A_1S_{n1}\widetilde{X}_1S_{n1}B_1+A_2S_{n2}\widetilde{X}_2S_{n2}B_2+\cdots+A_lS_{nl}\widetilde{X}_lS_{nl}B_l=\widetilde{C}\\ B_1^{\mathrm{T}}S_{n1}\widetilde{X}_1S_{n1}A_1^{\mathrm{T}}+B_2^{\mathrm{T}}S_{n2}\widetilde{X}_2S_{n2}A_2^{\mathrm{T}}+\cdots+B_l^{\mathrm{T}}S_{nl}\widetilde{X}_lS_{nl}A_l^{\mathrm{T}}=\widetilde{C}^{\mathrm{T}}\end{cases} \tag{5}$$

$$\widetilde{X}_i=X_i-\overline{\overline{X}}_i,\ \widetilde{C}=C-\sum_{i=1}^{l}A_i\overline{\overline{X}}_iB_i\qquad i=1,2,\cdots,l$$

(3)　　(5)　　$[X_1^*,\ X_2^*,\ \cdots,\ X_l^*]$,　　2

$$[\hat{X}_1,\ \hat{X}_2\cdots\hat{X}_l]=[\widetilde{X}_1^*+\overline{\overline{X}}_1,\ \widetilde{X}_2^*+\overline{\overline{X}}_2\cdots\widetilde{X}_l^*+\overline{\overline{X}}_l]$$

4　数值实验

$n=2$,　　A_1, B_1, A_2, B_2, C　　.

$$A_1=\begin{bmatrix}3&-5&3&-5&-2&3\\4&3&-1&3&-1&4\\3&-5&3&-5&-2&3\\4&3&-1&3&-1&4\\3&-5&3&-5&2&3\\4&3&-1&3&-1&4\end{bmatrix}$$

$$B_1=\begin{bmatrix}3&9&3&9&3&9\\7&4&7&4&7&4\\4&3&4&3&4&3\\-5&4&-5&4&-5&4\\3&4&3&4&3&4\\4&-5&4&-5&4&-5\end{bmatrix}$$

$$A_2=\begin{bmatrix}4&3&-1&3&-1&4\\3&-5&3&-5&2&5\\4&3&-1&3&-1&4\\3&-5&3&-5&2&5\\4&3&-1&3&-1&4\\3&-5&3&-5&2&5\end{bmatrix}$$

$$B_2=\begin{bmatrix}3&-4&3&-4&3&-4\\5&-3&5&-3&5&-3\\1&4&1&4&1&4\\-5&2&-5&2&-5&2\\-3&4&-3&4&-3&4\\4&-3&4&-3&4&-3\end{bmatrix}$$

$$C=\begin{bmatrix}68&91&54&77&51&83\\81&72&81&72&81&72\\68&91&54&77&51&83\\81&72&81&72&81&72\\68&91&54&77&51&83\\81&72&81&72&81&72\end{bmatrix}$$

$$\overline{\boldsymbol{X}}_1=\begin{bmatrix} -1 & 2 & 1 & -2 & 1 & 5 \\ 2 & 0 & 1 & 0 & -1 & 1 \\ 1 & 4 & 0 & -1 & 5 & 2 \\ 2 & 1 & -3 & 1 & 0 & 1 \\ 3 & -1 & 0 & -1 & 0 & 2 \\ 2 & -1 & -2 & 1 & 2 & -1 \end{bmatrix}$$

$$\overline{\boldsymbol{X}}_2=\begin{bmatrix} -2 & 0 & 3 & -2 & 1 & 1 \\ 0 & -1 & 0 & -2 & 3 & -1 \\ 3 & 2 & -1 & 2 & 0 & 1 \\ 3 & 1 & 2 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 & 5 & 3 \\ 1 & 4 & -2 & 1 & 5 & -2 \end{bmatrix}$$

1) 1 :

$$\boldsymbol{X}_1^{(40)}=\begin{bmatrix} 0.7319 & 0.5058 & 0.3761 & 0.0664 & 0.4632 & 0.7319 \\ 0.5058 & -0.3734 & -0.0234 & -0.1392 & -0.43498 & 0.4632 \\ 0.3761 & -0.0234 & 0.0823 & -0.0980 & -0.1392 & 0.0664 \\ 0.0664 & -0.1392 & -0.0980 & 0.0823 & -0.0234 & 0.3761 \\ 0.4632 & -0.4349 & -0.1392 & -0.0234 & -0.3734 & 0.5058 \\ 0.7319 & 0.4632 & 0.0664 & 0.3761 & 0.5058 & 0.7319 \end{bmatrix}$$

$$\overline{\boldsymbol{X}}_2^{(40)}=\begin{bmatrix} -0.0476 & 0.1135 & 0.0313 & 0.1517 & 0.0930 & -0.0869 \\ 0.1135 & -0.0751 & -0.0369 & 0.0371 & 0.0399 & 0.0930 \\ 0.0313 & -0.0369 & 0.1889 & -0.1639 & 0.0371 & 0.1517 \\ 0.1517 & 0.0371 & -0.1639 & 0.1889 & -0.0751 & 0.1135 \\ 0.0930 & 0.0399 & 0.0371 & -0.0369 & -0.0751 & 0.1135 \\ -0.0869 & 0.0930 & 0.1517 & 0.0313 & 0.1135 & -0.0476 \end{bmatrix}$$

$$\|\boldsymbol{R}^{(40)}\|=5.539\times10^{-11}, \quad \|\boldsymbol{A}_1\boldsymbol{X}_1^{(40)}\boldsymbol{B}_1+\boldsymbol{A}_2\boldsymbol{X}_2^{(40)}\boldsymbol{B}_2-\boldsymbol{C}\|^2=28.1069$$

2) 1, 2 :

$$\overline{\boldsymbol{X}}_1=\begin{bmatrix} -1.6974 & 1.3851 & 0.8018 & -0.4814 & 0.9160 & 2.8026 \\ 1.3851 & 1.0136 & 1.0895 & 1.5305 & -0.3492 & 0.9160 \\ 0.8018 & 1.0895 & -0.5111 & -1.2573 & 1.5305 & -0.4814 \\ -0.4814 & 1.5305 & -1.2573 & -0.5111 & 1.0895 & 0.8018 \\ 0.9160 & -0.3492 & 1.5305 & 1.0895 & 1.0136 & 1.3851 \\ 2.8026 & 0.9160 & -0.4814 & 0.8018 & 1.3851 & -1.6974 \end{bmatrix}$$

$$\overline{\boldsymbol{X}}_2=\begin{bmatrix} -1.4699 & 1.5668 & 1.5695 & -0.4404 & 1.2922 & 1.5626 \\ 1.5668 & 0.9683 & 1.0444 & -0.2963 & 1.9599 & 1.2922 \\ 1.5695 & 1.0444 & -1.0309 & 1.9353 & -0.2963 & -0.4404 \\ -0.4404 & -0.2963 & 1.9353 & -1.0309 & 1.0444 & 1.5695 \\ 1.2922 & 1.9599 & -0.2963 & 1.0444 & 0.9683 & 1.5668 \\ 1.5626 & 1.2922 & -0.4404 & 1.5695 & 1.5668 & -1.4699 \end{bmatrix}$$

$$\min_{(X,Y)\in S_E}\{\|\boldsymbol{X}_1-\overline{\boldsymbol{X}}_1\|^2+\|\boldsymbol{X}_2-\overline{\boldsymbol{X}}_2\|^2\}=18.4280$$

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The Least Squares Bisymmetric Solution Group of the Coupled Matrix Equation and Its Optimal Approximation

LIU Li, WANG Wei

School of Mathematics and Computer Science, Ningxia University, Yinchuan 750021, China

Abstract: Owing to the difficulty of obtaining the bisymmetric solution of a coupled matrix equation by matrix decomposition, an iterative method is proposed in this paper to solve the least squares bisymmetric solution group of the coupled matrix equation, and its convergence is proved. Its least-norm bisymmetric solution group can also be obtained by choosing the special kind of initial iterative matrices. In addition, the unique optimal approximation solution group to the given matrix group in Frobenius norm can be obtained.

Key words: matrix equation; least squares bisymmetric solution group; least norm solution group; optimal approximation solution group

责任编辑　张　构