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Boundedness of the Solution to the Nonlinear Kirchhoff Equation①

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Abstract: In this paper, by Galerkin approximation, the existence of solution for nonlinear Kirchhoff equation is proved. Moreover, the boundedness of solutions and the existence of bounded absorbing set are obtained.

Key words: boundedness of soultion; bounded absorbing set; Nonlinear Kirchhoff equation; Galerkin approximation

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In this paper, the main purpose of this work is to study the asymptotic behavior and existence of the solutions of a nonlinear Kirchhoff equation. To formalize this problem let us take Ω an open bounded set of R^n with smooth boundary Γ. Let us denote the unit normal vector by $\nu(x)$ with $x \in \Gamma$ outside of Ω and consider the following initial boundary value problem:

$$u_{tt} - M(\|\nabla u\|^2)\Delta u + \beta u_t + g(u) = f(x) \text{ in } Q = \Omega \times (0, \infty) \tag{1}$$

$$u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) \text{ in } x \in \Omega \tag{2}$$

$$u = 0 \text{ in } \Sigma = \Gamma \times (0, \infty) \tag{3}$$

here $\|\nabla u\|^2 = \sum_{i=1}^{n}\int_\Omega \left|\frac{\partial u}{\partial x_i}(x)\right|^2 dx$, $\Delta u = \sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}$, u is the transverse displacement. The function $g \in C^1$ satisfying the following conditions:

$$\liminf_{|s|\to\infty}\frac{G(s)}{s^2} \geqslant 0,\ G(s) = \int_0^s g(r)dr \tag{4}$$

$$\limsup_{|s|\to\infty}\frac{|g'(s)|}{|s|^r} = 0 \tag{5}$$

where $0 \leqslant \gamma < \infty (n = 1, 2)$, $0 \leqslant \gamma < 2 (n = 3)$, $\gamma = 0 (n \geqslant 4)$. Furthermore, there exists $C_1 > 0$ such that

$$\liminf_{|s|\to\infty}\frac{sg(s) - c_1 G(s)}{s^2} \geqslant 0 \tag{6}$$

For example, when $g(u) = |u|^\gamma u$, then (4)-(6) hold with $C_1 = 1 + \gamma$. We shall assume the function $M \in C^1([0, \infty))$ satisfies

$$M(\lambda) \geqslant m_0 > 0,\ |M'(\lambda)| \leqslant m'_0,\ m(\lambda)\lambda \geqslant \widehat{M}(\lambda),\ \forall \lambda \geqslant 0 \tag{7}$$

where $\widehat{M}(\lambda) = \int_0^\lambda M(s)ds$. Problems (1)-(3) have their origins in the mathematical description of small

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amplitude vibrations of an elastic string [1]. In fact, a mathematical model for the deflection of an elastic string of length $L > 0$ is given by the mixed problem for the nonlinear wave equation

$$\rho h \frac{\partial^2 u}{\partial t^2} = \left\{ p_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x \right\} \frac{\partial^2 u}{\partial x^2} \text{ for } 0 < x < L,\ t \geqslant 0 \tag{8}$$

where u is the lateral deflection, x is the space coordinate, t is the time, E is the Young's modulus, ρ is the mass density, h is the cross section area and ρ_0 is the initial axial tension. Kirchhoff was the first one to introduce (8) in the study of oscillations of stretched strings and plates, so that (8) is called the wave equation of Kirchhoff type after him. The existence of global solutions and exponential decay to Kirchhoff equation (1) has been investigated by many authors [2-3].

Throughout this paper we define $H(\Omega) = L^2(\Omega)$, $(u, v) = \int_\Omega u(x)v(x)\mathrm{d}x$, $\forall u, v \in L^2(\Omega)$, and $\| u \| = \| u \|_{L^2(\Omega)}$. We also set $V = H^1(\Omega)$, $E_0 = H^1(\Omega) \times H(\Omega)$.

1 Main Result and Proof

Here is the main result of the paper:

Theorem Assume that $(u_0, u_1) \in E_0$ and $f(x) \in H$, then the problem (1)-(3) admits a unique global solution u satisfying

$$u \in C(0, +\infty; H^1(\Omega)),\ u_t \in C(0, +\infty; H(\Omega))$$

Moreover, there exist $t_0 > 0$ and a constant $\rho_0 > 0$ such that

$$\| \nabla u(t) \|^2 + \| u_t(t) \|^2 \leqslant \rho_0^2, \text{ for } t \geqslant t_0$$

Proof We apply the methods of Galerkin approximation to equation(1) which the key step is priori estimates[4]. By (4), (6), and apply the Poincar′ e inequality, there exist constants K_1, $K_2 > 0$ only depending on u, such that

$$\int_\Omega G(u(x))\mathrm{d}x + \eta \| \nabla u \|^2 + K_1 \geqslant 0 \qquad \forall u \in V \tag{9}$$

$$\int_\Omega u g(u)\mathrm{d}x - c_1 \int_\Omega G(u(x))\mathrm{d}x + \eta \| \nabla u \|^2 + K_2 \geqslant 0 \qquad \forall u \in V \tag{10}$$

We take the scalar product in H of equation (1) with $\tilde{u} = u_t + \alpha u$, $0 < \alpha \leqslant \alpha_0$, α_0 will be chosen later. After a computation, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\| \tilde{u} \|^2 + \widehat{M}(\| \nabla u \|^2) + 2\int_\Omega G(u)\mathrm{d}x \right) + \alpha M(\| \nabla u \|^2)\| \nabla u \|^2$$
$$+ (\beta - \alpha)\| \tilde{u} \|^2 - \alpha(\beta - \alpha)\int_\Omega u\tilde{u}\,\mathrm{d}x + \alpha\int_\Omega g(u)u\mathrm{d}x = \int_\Omega f\tilde{u}\,\mathrm{d}x \tag{11}$$

We note that

$$(\beta - \alpha)\| \tilde{u} \|^2 - \alpha(\beta - \alpha)\int_\Omega u\tilde{u}\,\mathrm{d}x$$
$$\geqslant \left(\beta - \alpha - \frac{\alpha\beta^2}{\lambda_1 m_0}\right)\| \tilde{u} \|^2 - \frac{\alpha}{4}\widehat{M}(\| \nabla u \|^2)$$
$$\alpha\int_\Omega g(u)u\mathrm{d}x \geqslant \alpha C_1\int_\Omega G(u)\mathrm{d}x - \frac{a m_0}{4}\| \nabla u \|^2 - \alpha K_2(m_0)$$
$$\geqslant \alpha C_1\int_\Omega G(u)\mathrm{d}x - \frac{\alpha}{4}\widehat{M}(\| \nabla u \|^2) - \alpha K_2(m_0)$$

Thus we have

$$\frac{1}{2}\frac{d}{dt}\left(\| \tilde{u} \|^2 + \widehat{M}(\| \nabla u \|^2) + 2\int_\Omega G(u)\mathrm{d}x \right) + (\beta - \alpha - \frac{\alpha\beta^2}{\lambda_1 m_0})\| \tilde{u} \|^2$$
$$+ \frac{\alpha}{2}\widehat{M}(\| \nabla u \|^2) + \alpha C_1\int_\Omega G(u)\mathrm{d}x \leqslant \alpha K_2(m_0) + \int_\Omega f\tilde{u}\,\mathrm{d}x \tag{12}$$

Choose α_0 such that $\alpha_0\left(1+\frac{\beta^2}{\lambda_1 m_0}\right)=\frac{\beta}{2}$ Since $1+\frac{\beta^2}{\lambda_1 m_0}>1$, and $0<\alpha\leqslant\alpha_0$, we have

$$\beta-\alpha-\frac{\alpha\beta^2}{\lambda_1 m_0}\geqslant\frac{\beta}{2}\geqslant\alpha_0\geqslant\alpha$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\tilde{u}\|^2+\widehat{M}(\|\nabla u\|^2)+2\int_\Omega G(u)\mathrm{d}x\right)+\alpha(\|\tilde{u}\|^2$$
$$+\widehat{M}(\|\nabla u\|^2)+2C_1\int_\Omega G(u)\mathrm{d}x)\leqslant 2\alpha K_2(m_0)+\frac{1}{\alpha}\|f\|^2$$

Set $\delta=\min(\alpha, \alpha C_1)$ and then

$$\frac{d}{dt}\left(\|\tilde{u}\|^2+\widehat{M}(\|\nabla u\|^2)+2\int_\Omega G(u)\mathrm{d}x+2K_1\right)+\delta(\|\tilde{u}\|^2$$
$$+\widehat{M}(\|\nabla u\|^2)+2\int_\Omega G(u)\mathrm{d}x+2K_1)\leqslant 2\alpha K_2(m_0)+\frac{1}{\alpha}\|f\|^2+2\delta K_1 \tag{13}$$

Set

$$y(t)=\|\tilde{u}\|^2+\widehat{M}(\|\nabla u\|^2)+2\int_\Omega G(u)\mathrm{d}x+2K_1$$

From condition (9) we know $y(t)\geqslant 0$ and (13) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t)+\delta y(t)\leqslant 2\alpha K_2(m_0)+\frac{1}{\alpha}\|f\|^2+2\delta K_1$$

By Gronwall's inequality, we obtain

$$y(t)\leqslant y(0)e^{-\delta t}+\left(\frac{2\alpha K_2(m_0)}{\delta}+\frac{1}{\alpha\delta}\|f\|^2+2K_1\right)(1-e^{-\delta t}),\ t\geqslant 0 \tag{14}$$

For any bounded subset B of E_0, $(u_0, u_1)\in B$, $\widehat{M}(\|\nabla u\|^2)$ and $\int_\Omega G(u_0)\mathrm{d}x$ are bounded too. Hence

$$\begin{aligned}R=R(B)&=\sup_{(u_0,\,u_1)\in B}y(0)\\&=\sup_{(u_0,\,u_1)\in B}\left\{\|u_1+\alpha u_0\|^2+\widehat{M}(\|\nabla u_0\|^2)+2\int_\Omega G(u_0)\mathrm{d}x+2K_1\right\}<\infty\end{aligned}$$

and

$$\lim_{t\to\infty}\sup_{(u_0,\,u_1)\in B}y(t)\leqslant\frac{2\alpha K_2(m_0)}{\alpha}+\frac{1}{\alpha\delta}\|f\|^2+2K_1\equiv\mu_0^2 \tag{15}$$

Let $\mu_1>\mu_0$ be fixed, and

$$t_0=t_0(R, \mu_1)=\frac{1}{\alpha}\ln\frac{R}{\mu_1^2-\mu_0^2}$$

for any $t\geqslant t_0$, we have $y(t)\leqslant\mu_1^2$ and

$$\begin{aligned}\|\nabla u(t)\|^2+\|\nabla u_t(t)\|^2&=\|\nabla u(t)\|^2+\|u_t(t)+\alpha u(t)-\alpha u(t)\|^2\\&\leqslant\|\nabla u(t)\|^2+2(\|u_t(t)+\alpha u(t)\|^2+\alpha^2\|u(t)\|^2)\\&\leqslant\|\nabla u(t)\|^2+2(\|u_t(t)+\alpha u(t)\|^2+\frac{2\alpha^2}{\lambda_1})\|\nabla u(t)\|^2\\&\leqslant\left(\frac{1}{m_0}+\frac{2\alpha^2}{m_0\lambda_1}\right)\widehat{M}(\|\nabla u(t)\|^2)+2\|u_t(t)+\alpha u(t)\|^2\\&\leqslant\max\left\{\frac{1}{m_0}+\frac{2\alpha^2}{m_0\lambda_1}, 2\right\}(\widehat{M}(\|\nabla u(t)\|^2)+\|u_t(t)+\alpha u(t)\|^2)\\&\leqslant\max\left\{\frac{1}{m_0}+\frac{2\alpha^2}{m_0\lambda_1}, 2\right\}y(t)\\&\leqslant\max\left\{\frac{1}{m_0}+\frac{2\alpha^2}{m_0\lambda_1}, 2\right\}\mu_1^2\end{aligned}$$

where λ_1 is the first eigenvalue of $-\Delta$, $\lambda_1 = \inf_{v \in H^1,\ v \neq 0} \frac{\| \nabla v \|^2}{\| v \|^2}$. Thus we obtain

$$\| \nabla u(t) \|^2 + \| u_t(t) \|^2 \leqslant \rho_0^2 \qquad t \geqslant t_0 \tag{16}$$

where $\rho_0^2 = \max\left\{\frac{1}{m_0} + \frac{2\alpha^2}{m_0 \lambda_1},\ 2\right\}\mu_1^2$. The proof is completed.

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Kirchhoff 方程解的有界性

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摘要: 利用 Galerkin 方法证明了非线性 Kirchhoff 方程解的存在性，进一步得到了解的有界性和有界吸收集的存在性.

关键词: 解的有界性；界吸收集；非线性 Kirchhoff 方程；Galerkin 方法

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