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Special Projectively Flat (α, β)-Metrics①

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Abstract: In this paper, the authors study approximate exponential metric and prove that the second approximate exponential metric $F=\alpha+\beta+\frac{\beta^2}{2\alpha}+\frac{\beta^3}{6\alpha^2}$ is locally projectively flat if and only if α is projectively flat and β is parallel with respect to α, where $\alpha=\sqrt{a_{ij}y^iy^j}$, $\beta=b_iy^i$. And the same result is obtained for the higher order approximate exponential metric.

Key words: (α, β)-metric; projectively flat Finsler metric; exponential metric; approximate exponential metric
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Hilbert Fourth Problem is one of important problems in Finsler geometry, which is to study and characterize locally projectively Finsler metrics. It is important and interesting to study some projectively flat Finsler metric. Some special projectively flat (α, β)-metrics have been studied. The main purpose of this paper is to study and characterize locally projectively flat approximate exponential metric. As we all know, the exponential metric

$$F=\alpha e^{\frac{\beta}{\alpha}} \qquad \alpha=\sqrt{a_{ij}y^iy^j} \qquad \beta=b_iy^i \tag{1}$$

is an important metric in Finsler geometry. In the exponential metric, the 1-form $\beta=b_iy^i$ was originally to be induced by earth gravity. Hence we could regard b_i as the infinitesimals. Then it induces the approximate exponential metric if we neglect all powers not less than k of b_i for some positive integer k. It can be expressed as the form

$$F_\infty=\sum_{k=0}^{\infty}\frac{\beta^k}{\alpha^{k-1}k!} \tag{2}$$

where k is natural number and zero.

We regard b_i as very small numerically. If we neglect all the power not less than 2 of b_i in(2), then $F=\alpha+\beta$ is a Randers metric. In this case, there are non-trivial projectively flat metrics (i. e. α is not locally projectively flat and β is not parallel[1]. If we neglect all the powers not less than 3 of b_i in (2), then $F_2=\alpha+\beta+\frac{\beta^2}{2\alpha}+\frac{\beta^3}{6\alpha^2}$ is called the first approximate exponential metric. It is a special case in [1] and also are non-trivial projectively flat metrics. Here, we neglect all the power not less than 4 of b_i in(2), then

$$F=\alpha+\beta+\frac{\beta^2}{2\alpha}+\frac{\beta^3}{6\alpha^2} \tag{3}$$

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is an approximate metric to the exponential metric called the second approximate exponential metric.

Theorem 1 The second approximate exponential metric $F=\alpha+\beta+\frac{\beta^2}{2\alpha}+\frac{\beta^3}{6\alpha^2}$, is locally projectively flat if and only if

(i) β is parallel with respect to α;

(ii) α is projectively flat i. e. of constant curvature.

Proof For $F=\alpha+\beta+\frac{\beta^2}{2\alpha}+\frac{\beta^3}{6\alpha^2}$, that is, $F=\alpha\phi(s)$, $\phi(s)=1+s+\frac{s^2}{2}+\frac{s^3}{6}$, $s=\frac{\beta}{\alpha}$. According to the relation of the geodesic coefficients G^i and G^i_α ([1]), that is,

$$G^i=G^i_\alpha+\alpha Qs^i_0+J\{-2Q\alpha s_0+r_{00}\}\frac{y^i}{\alpha}+H\{-2Q\alpha s_0+r_{00}\}\left\{b^i-s\frac{y^i}{\alpha}\right\} \tag{4}$$

where

$$Q=\frac{\phi'}{\phi-s\phi'}$$

$$J=\frac{\phi'(\phi-s\phi')}{2\phi(\phi-s\phi')+(b^2-s^2)\phi''}$$

$$H=\frac{\phi''}{2((\phi-s\phi')+(b^2-s^2))\phi''}$$

$$s=\beta/\alpha \qquad b=\|\beta_x\|_\alpha$$

and the condition that (α, β) -metric is projectively flat Finsler metric[1], we can easily get the following by using Maple programm when assume that F is projectively flat

$$\begin{aligned}&\left(1-\frac{s^3}{3}-\frac{s^2}{2}\right)\left(-\frac{8s^3}{3}-3s^2+2b^2s+2+2b^2\right)(a_ml\alpha^2-y_my_l)G^m_\alpha+\\&\alpha^3s_{l0}\left(1+s+\frac{s^2}{2}\right)\left(-\frac{8s^3}{3}-3s^2+2b^2s+2+2b^2\right)-\\&2\alpha^2s_0\left(1+s+\frac{s^2}{2}\right)(1+s)(b_l\alpha-sy_l)+\\&\alpha r_{00}(1+s)\left(1-\frac{s^3}{3}-\frac{s^2}{2}\right)(b_l\alpha-sy_l)=0\end{aligned} \tag{5}$$

Plugging $s=\frac{\beta}{\alpha}$ into the above equation, we note that the coefficients of α must be zero (because α^{even} is a polynomial in y^i). Then the above equation is equivalent to the following two equations

$$\begin{aligned}&\left[\frac{8}{9}\beta^6+\left(\frac{3}{2}-\frac{2b^2}{3}\right)\alpha^2\beta^4-(4+b^2)\alpha^4\beta^2+(2b^2+2)\alpha^6\right](b_m\alpha^2-y_m\beta)G^m_\alpha-\\&\alpha\left\{s_0\left[\frac{\alpha^2\beta^5}{3}+\frac{5\alpha^4\beta^3}{3}-2\alpha^6\beta\right]+r_{00}(b^2\alpha^2-\beta^2)\left(\frac{\beta^4}{3}+\frac{\alpha^2\beta^2}{2}-\alpha^4\right)\right\}=0\end{aligned} \tag{6}$$

and

$$\begin{aligned}&\left[\frac{7}{3}\alpha\beta^5-\left(\frac{10}{3}+\frac{5b^2}{3}\right)\alpha^3\beta^3+2b^2\alpha^5\beta\right](b_m\alpha^2-y_m\beta)G^m_\alpha-\\&\alpha^2\left\{s_0\left[\frac{25}{6}\alpha^3\beta^4-2\alpha^7-3\alpha^3\beta^4\right]+r_{00}(b^2\alpha^2-\beta^2)\left(\frac{5}{6}\alpha\beta^3-\alpha^3\beta\right)\right\}=0\end{aligned} \tag{7}$$

Eliminating $(b_m\alpha^2-y_m\beta)G^m_\alpha$ and α^2 from (6) and (7) and rewriting the form that is obtained, yields

$$A+B\alpha^2+C\alpha^4+D\alpha^6+E\alpha^8+F\alpha^{10}+G\alpha^{12}=0 \tag{8}$$

where

$$A=-\frac{\beta^{11}}{27}r_{00}$$

$$B=\frac{\beta^9}{27}b^2r_{00}-\frac{7\beta^{10}}{27}s_0+\frac{11\beta^9}{36}r_{00}$$

$$C = -\frac{5\beta^7}{6}r_{00} - \frac{11\beta^7}{36}b^2 r_{00} + \frac{2\beta^8}{9}b^2 s_0 + \frac{37\beta^8}{36}s_0$$

$$D = \frac{7\beta^5}{3}r_{00} + \frac{5\beta^5}{6}b^2 r_{00} - \frac{17\beta^6}{18}b^2 s_0 - \frac{34\beta^6}{9}s_0$$

$$E = -\frac{7\beta^3}{3}b^2 r_{00} - 2\beta^3 r_{00} + \frac{22\beta^4}{3}s_0 + 3\beta^4 b^2 s_0$$

$$F = -8\beta^2 s_0 + 2r_{00} - b\beta^2 b^2 s_0$$

$$G = 4b^2 s_0 + 4s_0 \tag{9}$$

From(8), we can know A can be divided by α^2. Note that β^5 is not divided by α^2, then there is a scalar function $\rho = \rho(x)$, such that $r_{00} = \rho\alpha^2$. Then A and B become

$$A = -\frac{\beta^{11}}{27}\rho\alpha^2$$

$$B = \frac{\beta^9}{27}b^2\rho\alpha^2 - \frac{7\beta^{10}}{27}s_0 + \frac{11\beta^9}{36}\rho\alpha^2 \tag{10}$$

Plugging them into (8), we obtain

$$\beta^{10}\left(-\frac{\beta}{27}\rho - \frac{7s_0}{27}\right) + \left(C + \frac{\beta^9}{27}\rho b^2 + \frac{11\beta^9}{36}\rho\right)\alpha^2 + D\alpha^4 + E\alpha^6 + F\alpha^8 + G\alpha^{10} = 0 \tag{11}$$

From (11), we know that $\beta^{10}\left(-\frac{\beta}{27}\rho - \frac{7s_0}{27}\right)$ can be divided by α^2 . Note that β^{10} is not divided by α^2 , then $-\frac{\beta}{27}\rho - \frac{7s_0}{27}$ can be divided by α^2 . It is impossible unless

$$\frac{\beta}{27}\rho + \frac{7s_0}{27} = 0 \tag{12}$$

That is

$$\frac{b_i}{27}\rho + \frac{7s_i}{27} = 0 \tag{13}$$

Contracting (13) with b_i yields $\frac{b^2}{27}\rho = 0$. Since $b^2 \neq 0$, then $\rho = 0$. From $r_{00} = \rho\,\alpha^2$ and (12), we obtain

$$r_{00} = 0 \qquad s_0 = 0 \tag{14}$$

Plugging (14) into the form of the condition that (α, β) -metric is projectively flat[1] yields

$$\left(\alpha^3 - \frac{\alpha\beta^2}{2} - \frac{\beta^3}{3}\right)(a_m l\alpha^2 - y_m y_l)G^m_\alpha + \alpha^4(\alpha^2 + \alpha\beta + \frac{\beta^2}{2})s_{l0} = 0 \tag{15}$$

The coefficient of α must be zero (note: α^{even} is a polynomial in y^i). Then the above equation equivalent to

$$\left(\alpha^3 - \frac{\alpha\beta^2}{2}\right)(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \alpha^5_{l0} = 0$$

$$-\frac{\beta^3}{3}(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \alpha^6 s_{l0} + \frac{\beta^2}{2}\alpha^4 s_{l0} = 0 \tag{16}$$

From the above equation we can see $s_{l0} = 0$. By (14), β is parallel with respect to α . Plugging $s_{l0} = 0$ and $r_{00} = 0$ into (4) yields

$$G^i = G^i_\alpha \tag{17}$$

Since F is projectively flat, then there is a function $P = P(x, y)$ such that $G^i = Py^i$, where P is homogeneous in y of degree one. Then G^i_α , that is , α is projectively flat. On the other hand, it is clear that the converse is also true by the condition that (α, β) -metric is projectively flat[1]. This completes the proof.

Furthermore, for the higher order approximate exponential metric, we obtain the following

Theorem 2 The $(k-1)-th$ $(k \geqslant 3)$ approximate exponential metric $F_\infty = \sum_{k=0}^{\infty}\frac{\beta^k}{\alpha^{k-1}k!}$, is locally projectively flat if and only if

(i) β is parallel with respect to α ;

(ii) α is projectively flat, i. e. of constant curvature.

Proof Assume F is projectively flat, by the condition that (α, β) -metric is projectively flat Finsler metric[1], we obtain

$$2\left(1-\sum_{k=2}^{n}\frac{(k-1)s^k}{k!}\right)\left\{\left(1-\sum_{k=2}^{n}\frac{(k-1)s^k}{k!}\right)+(b^2-s^2)\sum_{k=2}^{n}\frac{s^k}{k!}\right\}(a_{ml}\alpha^2-y_my_l)G_\alpha^m+$$
$$2\alpha^3\sum_{k=0}^{n-1}\frac{s^k}{k!}\left\{\left(1-\sum_{k=2}^{n}\frac{(k-1)s^k}{k!}\right)+(b^2-s^2)\sum_{k=2}^{n}\frac{s^k}{k!}\right\}s_{l0}+\left(-2\alpha^2\sum_{k=0}^{n-1}\frac{s^k}{k!}\sum_{k=2}^{n}\frac{s^k}{k!}\right)(b_l\alpha-sy_l)s_0+$$
$$\alpha\left(1-\sum_{k=2}^{n}\frac{(k-1)s^k}{k!}\right)\sum_{k=2}^{n}\frac{s^k}{k!}(b_l\alpha-sy_l)r_{00}=0 \tag{18}$$

When k is an odd number, let

$$X^0=1-\sum_{i=1}^{k/2}\frac{s^{2i-1}}{(2i)!}s^{2i}$$
$$X^1=-\sum_{j=1}^{k/2}\frac{2j}{(2j+1)!}s^{2j+1}$$
$$Y^0=1+\sum_{i=1}^{k/2}\left[b^2\frac{s^{2i-2}}{(2i-2)!}+\frac{(2i-1)^2}{(2i)!}s^{2i}\right]$$
$$Y^1=\sum_{j=1}^{(k-2)/2}\left[b^2\frac{s^{2j+1}}{(2j+1)!}+\frac{(2j+2)^2}{(2j+3)}s^{2j+3}\right]$$
$$Z^0=\sum_{i=1}^{k/2}s^2\frac{i-1}{(2i-2)!}$$
$$Z^1=-\sum_{j=1}^{k/2}\frac{s^{2j-1}}{(2j-1)!}$$
$$W^0=\sum_{i=1}^{k/2}\frac{s^{2i-2}}{(2i-2)!}$$
$$W^1=-\sum_{j=1}^{(k-2)/2}\frac{s^{2j-1}}{(2j-1)!} \tag{19}$$

Thus

$$X^0+X^1=1-\sum_{k=2}^{n}\frac{k-1}{k!}s^k$$
$$Y^0+Y^1=1+\sum_{k=2}^{n}\left[b^2\frac{s^{k-2}}{(k-2)!}+\frac{(k-1)^2}{k!}s^k\right]$$
$$Z^0+Z^1=\sum_{k=1}^{n}\frac{s^{k-1}}{(k-1)!}$$
$$W^0+W^1=\sum_{k=2}^{n}\frac{s^{k-2}}{(k-2)!} \tag{20}$$

Then we obtain

$$2\alpha^{2k}(X^0Y^0+X^1Y^1)(b_m\alpha^2-y_m\beta)G_\alpha^m+2\alpha^{2k+3}(Z^0Y^1+Z^1Y^0)s_0-$$
$$2\alpha\ 2k+1(W^0Z^1+W^1Z^0)(b^2\alpha^2-\beta^2)s_0+\alpha^{2k}(W^0X^0+W^1X^1)(b^2\alpha^2-\beta^2)r_{00}=0 \tag{21}$$

and

$$2\alpha^{2k}(X^1Y^0+X^0Y^1)(b_m\alpha^2-y_m\beta)G_\alpha^m+2\alpha^{2k+3}(Z^0Y^0+Z^1Y^1)s_0-$$
$$2\alpha\ 2k+1(W^0Z^0+W^1Z^1)(b^2\alpha^2-\beta^2)s_0+\alpha^{2k}(W^0X^1+W^1X^0)(b^2\alpha^2-\beta^2)r_{00}=0 \tag{22}$$

Now with the same discussion in Theorem 1, we obtain $s_0=0$, $(a_{ml}\alpha^2-y_my_l)G_\alpha^m=0$, and $s_{l0}=0$. Thus β is parallel with respect to α and α is locally projective flat. When k is an odd number, the proof is similar.

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特殊的射影平坦(α，β)-度量

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摘要：研究了近似指数度量并得到二阶近似指数度量射影平坦的充要条件是 α 射影平坦，β 关于 α 平行. 且对高阶指数度量也得到了相同的结果. 这里，$\alpha=\sqrt{a_{ij}y^iy^j}$，$\beta=b_iy^i$.

关键词：(α，β)-度量；射影平坦芬斯勒度量；指数度量；近似指数度量

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