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非线性中立型微分方程的振动准则[①]

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摘要: 讨论一类二阶非线性中立型微分方程, 通过引入参数函数, 结合完全平方技术, 给出了该类方程解振动的判别准则.

关 键 词: 非线性; 中立型微分方程; 振动

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近年来, 中立型泛函微分方程的振动理论得到了很大的发展, 已有许多研究成果[1-4].

最近, 文献[5]讨论了二阶非线性中立型微分方程

$$[x(t)+p(t)x(t-\tau)]''+q(t)f(x(g_1(t)),\ x(g_2(t)))=0 \qquad t\geqslant t_0>0$$

的解的振动性.

本文考虑如下更广泛的二阶非线性中立型微分方程

$$[\alpha(t)(x(t)+p(t)x(t-\tau))']'+F(t,\ x(g_1(t)),\ x(g_2(t)),\ \cdots,\ x(g_m(t)))=0 \qquad t\geqslant t_0>0 \tag{1}$$

其中 $\tau>0$ 为常数. 通过引入参数函数, 结合完全平方技术, 给出方程(1)解振动的判别准则, 所得结果推广了已有文献的部分结果.

关于方程(1), 本文始终假设下列条件成立:

(H_1) $a(t)\in C([t_0,\ \infty),\ (0,\ \infty))$, $p(t)\in C([t_0,\ \infty),\ [0,\ \infty))$;

(H_2) $F(t,\ u_1,\ u_2,\ \cdots,\ u_m)\in C(R^{m+1},\ R)$, 且存在 $f(u_1,\ u_2,\ \cdots,\ u_m)\in C(R^m,\ R)$, 当 $u_i>0$ (或 $u_i<0$) 时, $u_if(u_1,\ u_2,\ \cdots,\ u_m)>0$ ($i\in I_m=\{1,\ 2,\ \cdots,\ m\}$), $f(u_1,\ u_2,\ \cdots,\ u_m)$ 关于 $u_1,u_2,\cdots,u_m$ 非增, 且存在常数 $c>0$ 及 $i_0\in I_m$, 使得

$$\left|\frac{f(u_1,\ u_2,\ \cdots,\ u_m)}{u_{i_0}}\right|\geqslant c \qquad u_{i_0}\neq 0$$

同时存在 $q(t)\in C([t_0,\ \infty),\ [0,\ \infty))$, 且 $q(t)$ 在任一区间 $[t_1,\ \infty)$ $(t_1\geqslant t_0)$ 上不恒为零, 使得当 $u_i\neq 0$ ($i\in I_m$) 时, $\dfrac{F(t,\ u_1,\ u_2,\ \cdots,\ u_m)}{f(u_1,\ u_2,\ \cdots,\ u_m)}\geqslant q(t)$;

(H_3) 存在函数 $\sigma(t)\in C'([t_0,\ \infty),\ (0,\ \infty))$, 使得 $\sigma(t)\leqslant g_i(t)\leqslant t$, 且 $\lim\limits_{t\to\infty}\sigma(t)=\infty$, $\sigma'(t)>0$, $i\in I_m$.

为得到本文的结果, 首先给出如下引理.

引理 若 $x(t)$ 是方程(1)的最终正解, 并且令

$$y(t)=x(t)+p(t)x(t-\tau) \tag{2}$$

则最终有

$$y(t)>0 \qquad y'(t)>0 \qquad [a(t)y'(t)]'\leqslant 0 \tag{3}$$

证 由 $x(t)$ 是方程(1)的最终正解及条件(H_3)可知, 存在充分大的 $t_1\geqslant t_0$, 使得

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$$x(t)>0 \qquad x(t-\tau)>0 \qquad x(g_i(t))>0 \qquad i\in I_m,\ t\geqslant t_1$$

再由(H_2)得

$$f(x(g_1(t)),x(g_2(t)),\cdots,x(g_m(t)))>0 \qquad t\geqslant t_1$$

于是由(2)，我们有

$$y(t)>0 \qquad t\geqslant t_1$$

由(1)及条件(H_2)，有

$$\begin{aligned}[a(t)y'(t)]' &= -F(t,x(g_1(t)),x(g_2(t)),\cdots,x(g_m(t)))\\ &\leqslant -q(t)f(x(g_1(t)),x(g_2(t)),\cdots,x(g_m(t)))\leqslant 0 \qquad t\geqslant t_1\end{aligned}$$

下面证明 $y'(t)>0$，$t\geqslant t_1$.

事实上，若存在 $t_2\geqslant t_1$，使得 $y'(t_2)\leqslant 0$，则当 $t\geqslant t_2$ 时，$y'(t)\leqslant y'(t_2)\leqslant 0$. 再由 $q(t)$ 不恒为零知，存在 $t_3\geqslant t_2$，使得 $y'(t_3)<0$，且有

$$y'(t)\leqslant y'(t_3) \qquad t\geqslant t_3$$

取定 $T\geqslant t_3$，并对上述不等式从 T 到 t 积分($t>T$)得

$$y(t)\leqslant y(T)+\int_T^t y'(t_3)\mathrm{d}s=y(T)+y'(t_3)(t-T)$$

令 $t\to\infty$，并注意到 $y'(t_3)<0$，有

$$\lim_{t\to\infty}y(t)=-\infty$$

此与 $y(t)>0$ 矛盾. 引理得证.

定理 1　假设条件(H_1)－(H_3)成立，并且

(H_4)存在函数 $H(t,s)\in C'(D,R)$，$h(t,s)\in C(D,R)$ 及 $\rho(s)\in C'([t_0,\infty),(0,\infty))$，其中 $D=\{(t,s)\mid t\geqslant s\geqslant t_0\}$ 满足

(ⅰ) $H(t,t)=0$，$t\geqslant t_0$；$H(t,s)>0$，$t>s\geqslant t_0$；

(ⅱ) $\dfrac{\partial H(t,s)}{\partial s}\leqslant 0$，$(t,s)\in D$，且

$$-\frac{\partial(H(t,s)\rho(s))}{\partial s}=h(t,s)\sqrt{H(t,s)\rho(s)}$$

若

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t\left[cH(t,s)\rho(s)q(s)-\frac{a(\sigma(s))h^2(t,s)}{4\sigma'(s)}\right]\mathrm{d}s=\infty \tag{4}$$

则方程(1)的所有解振动.

证　不失一般性，假设方程(1)存在最终正解 $x(t)$，则由引理可知，存在 $t_1\geqslant t_0$，使得当 $t\geqslant t_1$ 时

$$x(t)>0 \qquad x(t-\tau)>0 \qquad x(\sigma(t))>0 \qquad x(g_i(t))>0 \qquad i\in I_m$$
$$f(x(g_1(t)),x(g_2(t)),\cdots,x(g_m(t)))>0$$
$$y(t)>0 \qquad y'(t)>0 \qquad [a(t)y'(t)]'\leqslant 0$$

令

$$z(t)=\frac{a(t)y'(t)}{y(\sigma(t))} \tag{5}$$

则

$$\begin{aligned}z'(t)&=\frac{[a(t)y'(t)]'}{y(\sigma(t))}-\frac{a(t)y'(t)y'(\sigma(t))\sigma'(t)}{y^2(\sigma(t))}\\ &=\frac{-F(t,x(g_1(t)),x(g_2(t)),\cdots,x(g_m(t)))}{y(\sigma(t))}-z(t)\frac{y'(\sigma(t))\sigma'(t)}{y(\sigma(t))}\\ &\leqslant -q(t)\frac{f(x(g_1(t)),x(g_2(t)),\cdots,x(g_m(t)))}{y(\sigma(t))}-z(t)\frac{y'(\sigma(t))\sigma'(t)}{y(\sigma(t))}\end{aligned} \tag{6}$$

由(2)得 $x(t)\leqslant y(t)$，于是 $x(g_i(t))\leqslant y(g_i(t))$ $(i\in I_m)$，注意到 $y'(t)>0$ 及条件(H_2)与(H_3)，我们有

$$\frac{f(x(g_1(t)),x(g_2(t)),\cdots,x(g_m(t)))}{y(\sigma(t))}\geqslant\frac{f(y(g_1(t)),y(g_2(t)),\cdots,y(g_m(t)))}{y(g_i(t))}\geqslant c \tag{7}$$

又由$[a(t)y'(t)]'\leqslant 0$及(H_3)，并注意到$z(t)>0$，我们有

$$\begin{aligned}z(t)\frac{y'(\sigma(t))\sigma'(t)}{y(\sigma(t))}&=z(t)\frac{[a(\sigma(t))y'(\sigma(t))]\sigma'(t)}{y(\sigma(t))a(\sigma(t))}\\&\geqslant z(t)\frac{[a(t)y'(t)]\sigma'(t)}{y(\sigma(t))a(\sigma(t))}=z^2(t)\frac{\sigma'(t)}{a(\sigma(t))}\end{aligned}\tag{8}$$

由(6),(7)和(8)得

$$z'(t)\leqslant -cq(t)-\frac{\sigma'(t)}{a(\sigma(t))}z^2(t)\tag{9}$$

由此可得

$$cq(t)\leqslant -z'(t)-\frac{\sigma'(t)}{a(\sigma(t))}z^2(t)\tag{10}$$

于是，对于任意的$t>t_2\geqslant t_1$，有

$$\begin{aligned}&\int_{t_2}^{t}cH(t,s)\rho(s)q(s)\mathrm{d}s\\&\leqslant -\int_{t_2}^{t}H(t,s)\rho(s)z'(s)\mathrm{d}s-\int_{t_2}^{t}H(t,s)\rho(s)\frac{\sigma'(s)}{a(\sigma(s))}z^2(s)\mathrm{d}s\\&=H(t,t_2)\rho(t_2)z(t_2)+\int_{t_2}^{t}\frac{\partial(H(t,s)\rho(s))}{\partial s}z(s)\mathrm{d}s-\int_{t_2}^{t}H(t,s)\rho(s)\frac{\sigma'(s)}{a(\sigma(s))}z^2(s)\mathrm{d}s\\&=H(t,t_2)\rho(t_2)z(t_2)-\int_{t_2}^{t}h(t,s)\sqrt{H(t,s)\rho(s)}z(s)\mathrm{d}s-\int_{t_2}^{t}H(t,s)\rho(s)\frac{\sigma'(s)}{a(\sigma(s))}z^2(s)\mathrm{d}s\\&=H(t,t_2)\rho(t_2)z(t_2)-\int_{t_2}^{t}\left[\sqrt{\frac{H(t,s)\rho(s)\sigma'(s)}{a(\sigma(s))}}z(s)+\frac{1}{2}h(t,s)\sqrt{\frac{a(\sigma(s))}{\sigma'(s)}}\right]^2\mathrm{d}s+\int_{t_2}^{t}\frac{a(\sigma(s))}{4\sigma'(s)}h^2(t,s)\mathrm{d}s\\&\leqslant H(t,t_2)\rho(t_2)z(t_2)+\int_{t_2}^{t}\frac{a(\sigma(s))}{4\sigma'(s)}h^2(t,s)\mathrm{d}s\end{aligned}$$

进一步有

$$\int_{t_2}^{t}\left[cH(t,s)\rho(s)q(s)-\frac{a(\sigma(s))}{4\sigma'(s)}h^2(t,s)\right]\mathrm{d}s\leqslant H(t,t_2)\rho(t_2)z(t_2)$$

由$\dfrac{\partial H(t,s)}{\partial s}\leqslant 0$，得$H(t,t_2)\leqslant H(t,t_0)(t_2\geqslant t_0)$，因此有

$$\int_{t_2}^{t}\left[cH(t,s)\rho(s)q(s)-\frac{a(\sigma(s))}{4\sigma'(s)}h^2(t,s)\right]\mathrm{d}s\leqslant H(t,t_0)\rho(t_2)z(t_2)$$

于是

$$\frac{1}{H(t,t_0)}\int_{t_2}^{t}\left[cH(t,s)\rho(s)q(s)-\frac{a(\sigma(s))}{4\sigma'(s)}h^2(t,s)\right]\mathrm{d}s\leqslant \rho(t_2)z(t_2)$$

进一步有

$$\begin{aligned}&\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[cH(t,s)\rho(s)q(s)-\frac{a(\sigma(s))}{4\sigma'(s)}h^2(t,s)\right]\mathrm{d}s\\&=\frac{1}{H(t,t_0)}\int_{t_0}^{t_2}\left[cH(t,s)\rho(s)q(s)-\frac{a(\sigma(s))}{4\sigma'(s)}h^2(t,s)\right]\mathrm{d}s+\frac{1}{H(t,t_0)}\int_{t_2}^{t}\left[cH(t,s)\rho(s)q(s)-\frac{a(\sigma(s))}{4\sigma'(s)}h^2(t,s)\right]\mathrm{d}s\\&\leqslant\int_{t_0}^{t_2}c\frac{H(t,s)}{H(t,t_0)}\rho(s)q(s)\mathrm{d}s+\rho(t_2)z(t_2)\\&\leqslant\int_{t_0}^{t_2}c\rho(s)q(s)\mathrm{d}s+\rho(t_2)z(t_2)\end{aligned}$$

令$t\to\infty$，并取上极限得

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[cH(t,s)\rho(s)q(s)-\frac{a(\sigma(s))}{4\sigma'(s)}h^2(t,s)\right]\mathrm{d}s<\infty$$

此与(4)矛盾. 定理1证毕.

由定理1的证明过程易得

推论 1　若定理 1 的(4) 式被替换为

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\rho(s)q(s)\mathrm{d}s=\infty \tag{11}$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{a(\sigma(s))}{\sigma'(s)}h^2(t,s)\mathrm{d}s<\infty \tag{12}$$

则方程(1) 的所有解振动.

定理 2　假设条件(H_1)－(H_3) 成立，如果存在常数 $m\geqslant 2$ 和函数 $\rho(s)\in C'([t_0,\infty),(0,\infty))$，使得

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_0}^{t}(t-s)^m\rho(s)q(s)\mathrm{d}s=\infty \tag{13}$$

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_0}^{t}\frac{a(\sigma(s))}{\sigma'(s)\rho(s)}(t-s)^{m-2}[m\rho(s)-(t-s)\rho'(s)]^2\mathrm{d}s<\infty \tag{14}$$

则方程(1) 的所有解振动.

证　不失一般性，假设方程(1) 存在最终正解 $x(t)$，则由定理 1 的证明可知存在 $t_1\geqslant t_0$，使得当 $t\geqslant t_1$ 时，有

$$cq(t)\leqslant -z'(t)-\frac{\sigma'(t)}{a(\sigma(t))}z^2(t)$$

于是，对于任意的 $t\geqslant t_1$，有

$$\begin{aligned}
&\int_{t_1}^{t}c(t-s)^m\rho(s)q(s)\mathrm{d}s\\
\leqslant&-\int_{t_1}^{t}(t-s)^m\rho(s)z'(s)\mathrm{d}s-\int_{t_1}^{t}(t-s)^m\rho(s)\frac{\sigma'(s)}{a(\sigma(s))}z^2(s)\mathrm{d}s\\
=&(t-t_1)^m\rho(t_1)z(t_1)+\int_{t_1}^{t}[-m(t-s)^{m-1}\rho(s)+(t-s)^m\rho'(s)]z(s)\mathrm{d}s-\int_{t_1}^{t}(t-s)^m\rho(s)\frac{\sigma'(s)}{a(\sigma(s))}z^2(s)\mathrm{d}s\\
=&(t-t_1)^m\rho(t_1)z(t_1)-\int_{t_1}^{t}(t-s)^{m-1}[m\rho(s)-(t-s)\rho'(s)]z(s)\mathrm{d}s-\int_{t_1}^{t}(t-s)^m\rho(s)\frac{\sigma'(s)}{a(\sigma(s))}z^2(s)\mathrm{d}s\\
=&(t-t_1)^m\rho(t_1)z(t_1)-\int_{t_1}^{t}\left\{\sqrt{\frac{\sigma'(s)}{a(\sigma(s))}\rho(s)}(t-s)^{\frac{m}{2}}z(s)+\frac{(t-s)^{\frac{m}{2}-1}[m\rho(s)-(t-s)\rho'(s)]}{2\sqrt{\frac{\sigma'(s)}{a(\sigma(s))}\rho(s)}}\right\}^2\mathrm{d}s+\\
&\int_{t_1}^{t}\frac{a(\sigma(s))(t-s)^{m-2}}{4\sigma'(s)\rho(s)}[m\rho(s)-(t-s)\rho'(s)]^2\mathrm{d}s\\
\leqslant&(t-t_1)^m\rho(t_1)z(t_1)+\int_{t_1}^{t}\frac{a(\sigma(s))(t-s)^{m-2}}{4\sigma'(s)\rho(s)}[m\rho(s)-(t-s)\rho'(s)]^2\mathrm{d}s
\end{aligned}\tag{15}$$

进一步有

$$\begin{aligned}
&\frac{1}{t^m}\int_{t_1}^{t}c(t-s)^m\rho(s)q(s)\mathrm{d}s\\
\leqslant&\frac{1}{t^m}(t-t_1)^m\rho(t_1)z(t_1)+\frac{1}{t^m}\int_{t_1}^{t}\frac{a(\sigma(s))(t-s)^{m-2}}{4\sigma'(s)\rho(s)}[m\rho(s)-(t-s)\rho'(s)]^2\mathrm{d}s
\end{aligned}$$

于是，有

$$\begin{aligned}
&\frac{1}{t^m}\int_{t_0}^{t}c(t-s)^m\rho(s)q(s)\mathrm{d}s\\
=&\frac{1}{t^m}\int_{t_0}^{t_1}c(t-s)^m\rho(s)q(s)\mathrm{d}s+\frac{1}{t^m}\int_{t_1}^{t}c(t-s)^m\rho(s)q(s)\mathrm{d}s\\
\leqslant&\frac{1}{t^m}\int_{t_0}^{t_1}c(t-s)^m\rho(s)q(s)\mathrm{d}s+\frac{1}{t^m}(t-t_1)^m\rho(t_1)z(t_1)+\frac{1}{t^m}\int_{t_1}^{t}\frac{a(\sigma(s))(t-s)^{m-2}}{4\sigma'(s)\rho(s)}[m\rho(s)-(t-s)\rho'(s)]^2\mathrm{d}s\\
\leqslant&\int_{t_0}^{t_1}c\left(1-\frac{s}{t}\right)^m\rho(s)q(s)\mathrm{d}s+\left(1-\frac{t_1}{t}\right)^m\rho(t_1)z(t_1)+\frac{1}{t^m}\int_{t_0}^{t}\frac{a(\sigma(s))(t-s)^{m-2}}{4\sigma'(s)\rho(s)}[m\rho(s)-(t-s)\rho'(s)]^2\mathrm{d}s
\end{aligned}$$

令 $t\to\infty$，并取上极限得

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_0}^{t}c(t-s)^m\rho(s)q(s)\mathrm{d}s$$

$$\leqslant\int_{t_0}^{t_1}c\rho(s)q(s)\mathrm{d}s+\rho(t_1)z(t_1)+\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_0}^{t}\frac{a(\sigma(s))(t-s)^{m-2}}{4\sigma'(s)\rho(s)}[m\rho(s)-(t-s)\rho'(s)]^2\mathrm{d}s<\infty$$

此与(13)矛盾. 定理 2 证毕.

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Oscillation Criteria for Nonlinear Neutral Differential Equations

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Abstract: A class of second order nonlinear neutral differential equations is discussed. Oscillatory criteria of the equations are obtained. The corresponding results in known literature are improved and extended.

Key words: nonlinear; neutral differential equation; oscillation

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