

# 非线性中立型微分方程的振动准则<sup>①</sup>

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**摘要:** 讨论一类二阶非线性中立型微分方程, 通过引入参数函数, 结合完全平方技术, 给出了该类方程解振动的判别准则.

**关键词:** 非线性; 中立型微分方程; 振动

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近年来, 中立型泛函微分方程的振动理论得到了很大的发展, 已有许多研究成果<sup>[1-4]</sup>.

最近, 文献[5]讨论了二阶非线性中立型微分方程

$$[x(t) + p(t)x(t-\tau)]'' + q(t)f(x(g_1(t)), x(g_2(t))) = 0 \quad t \geq t_0 > 0$$

的解的振动性.

本文考虑如下更广泛的二阶非线性中立型微分方程

$$[\alpha(t)(x(t) + p(t)x(t-\tau))]' + F(t, x(g_1(t)), x(g_2(t)), \dots, x(g_m(t))) = 0 \quad t \geq t_0 > 0 \quad (1)$$

其中  $\tau > 0$  为常数. 通过引入参数函数, 结合完全平方技术, 给出方程(1)解振动的判别准则, 所得结果推广了已有文献的部分结果.

关于方程(1), 本文始终假设下列条件成立:

(H<sub>1</sub>)  $a(t) \in C([t_0, \infty), (0, \infty))$ ,  $p(t) \in C([t_0, \infty), [0, \infty))$ ;

(H<sub>2</sub>)  $F(t, u_1, u_2, \dots, u_m) \in C(R^{m+1}, R)$ , 且存在  $f(u_1, u_2, \dots, u_m) \in C(R^m, R)$ , 当  $u_i > 0$  (或  $u_i < 0$ ) 时,  $u_i f(u_1, u_2, \dots, u_m) > 0$  ( $i \in I_m = \{1, 2, \dots, m\}$ ),  $f(u_1, u_2, \dots, u_m)$  关于  $u_1, u_2, \dots, u_m$  非增, 且存在常数  $c > 0$  及  $i_0 \in I_m$ , 使得

$$\left| \frac{f(u_1, u_2, \dots, u_m)}{u_{i_0}} \right| \geq c \quad u_{i_0} \neq 0$$

同时存在  $q(t) \in C([t_0, \infty), [0, \infty))$ , 且  $q(t)$  在任一区间  $[t_1, \infty)$  ( $t_1 \geq t_0$ ) 上不恒为零, 使得当  $u_i \neq 0$  ( $i \in I_m$ ) 时,  $\frac{F(t, u_1, u_2, \dots, u_m)}{f(u_1, u_2, \dots, u_m)} \geq q(t)$ ;

(H<sub>3</sub>) 存在函数  $\sigma(t) \in C'([t_0, \infty), (0, \infty))$ , 使得  $\sigma(t) \leq g_i(t) \leq t$ , 且  $\lim_{t \rightarrow \infty} \sigma(t) = \infty$ ,  $\sigma'(t) > 0$ ,  $i \in I_m$ .

为得到本文的结果, 首先给出如下引理.

**引理** 若  $x(t)$  是方程(1)的最终正解, 并且令

$$y(t) = x(t) + p(t)x(t-\tau) \quad (2)$$

则最终有

$$y(t) > 0 \quad y'(t) > 0 \quad [a(t)y'(t)]' \leq 0 \quad (3)$$

**证** 由  $x(t)$  是方程(1)的最终正解及条件(H<sub>3</sub>)可知, 存在充分大的  $t_1 \geq t_0$ , 使得

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$$x(t) > 0 \quad x(t - \tau) > 0 \quad x(g_i(t)) > 0 \quad i \in I_m, t \geq t_1$$

(H<sub>2</sub>) ^

$$f(x(g_1(t)), x(g_2(t)), \dots, x(g_m(t))) > 0 \quad t \geq t_1$$

(2), e

$$y(t) > 0 \quad t \geq t_1$$

(1) (H<sub>2</sub>), e

$$[a(t)y'(t)]' = -F(t, x(g_1(t)), x(g_2(t)), \dots, x(g_m(t))) \\ \leq -q(t)f(x(g_1(t)), x(g_2(t)), \dots, x(g_m(t))) \leq 0 \quad t \geq t_1$$

s y'(t) > 0, t ≥ t<sub>1</sub>.

, t<sub>2</sub> ≥ t<sub>1</sub>, ^ y'(t<sub>2</sub>) ≤ 0, X t ≥ t<sub>2</sub> , y'(t) ≤ y'(t<sub>2</sub>) ≤ 0. q(t) v , t<sub>3</sub> ≥ t<sub>2</sub>, ^ y'(t<sub>3</sub>) < 0, e

$$y'(t) \leq y'(t_3) \quad t \geq t_3$$

T ≥ t<sub>3</sub>, T - t ? (t > T) ^

$$y(t) \leq y(T) + \int_T^t y'(t_3) ds = y(T) + y'(t_3)(t - T)$$

t → ∞, - y'(t<sub>3</sub>) < 0, e

$$\lim_{t \rightarrow \infty} y(t) = -\infty$$

( y(t) > 0 . D] ^ .

定理 1 (H<sub>1</sub>) - (H<sub>3</sub>) j < ,

(H<sub>4</sub>) G' H(t, s) ∈ C'(D, R), h(t, s) ∈ C(D, R) ρ(s) ∈ C'([t<sub>0</sub>, ∞), (0, ∞)), u; D = {(t, s) | t ≥ s ≥ t<sub>0</sub>}

(i) H(t, t) = 0, t ≥ t<sub>0</sub>; H(t, s) > 0, t > s ≥ t<sub>0</sub>;

(ii) ∂H(t, s) / ∂s ≤ 0, (t, s) ∈ D,

$$-\frac{\partial(H(t, s)\rho(s))}{\partial s} = h(t, s)\sqrt{H(t, s)\rho(s)}$$

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left[ cH(t, s)\rho(s)q(s) - \frac{a(\sigma(s))h^2(t, s)}{4\sigma'(s)} \right] ds = \infty \tag{4}$$

X@A(1) T x e QRS.

证 4 : , @A(1) | ~ Q x(t), X D] , t<sub>1</sub> ≥ t<sub>0</sub>, ^ t ≥ t<sub>1</sub>

$$x(t) > 0 \quad x(t - \tau) > 0 \quad x(\sigma(t)) > 0 \quad x(g_i(t)) > 0 \quad i \in I_m$$

$$f(x(g_1(t)), x(g_2(t)), \dots, x(g_m(t))) > 0$$

$$y(t) > 0 \quad y'(t) > 0 \quad [a(t)y'(t)]' \leq 0$$

$$z(t) = \frac{a(t)y'(t)}{y(\sigma(t))} \tag{5}$$

X

$$z'(t) = \frac{[a(t)y'(t)]'}{y(\sigma(t))} - \frac{a(t)y'(t)y'(\sigma(t))\sigma'(t)}{y^2(\sigma(t))} \\ = \frac{-F(t, x(g_1(t)), x(g_2(t)), \dots, x(g_m(t)))}{y(\sigma(t))} - z(t) \frac{y'(\sigma(t))\sigma'(t)}{y(\sigma(t))} \\ \leq -q(t) \frac{f(x(g_1(t)), x(g_2(t)), \dots, x(g_m(t)))}{y(\sigma(t))} - z(t) \frac{y'(\sigma(t))\sigma'(t)}{y(\sigma(t))} \tag{6}$$

(2) ^ x(t) ≤ y(t), | x(g<sub>i</sub>(t)) ≤ y(g<sub>i</sub>(t)) (i ∈ I<sub>m</sub>), - y'(t) > 0 (H<sub>2</sub>) ( (H<sub>3</sub>), e

$$\frac{f(x(g_1(t)), x(g_2(t)), \dots, x(g_m(t)))}{y(\sigma(t))} \geq \frac{f(y(g_1(t)), y(g_2(t)), \dots, y(g_m(t)))}{y(g_i(t))} \geq c \tag{7}$$

又由  $[a(t)y'(t)]' \leq 0$  及  $(H_3)$ , 并注意到  $z(t) > 0$ , 我们有

$$\begin{aligned} z(t) \frac{y'(\sigma(t))\sigma'(t)}{y(\sigma(t))} &= z(t) \frac{[a(\sigma(t))y'(\sigma(t))]\sigma'(t)}{y(\sigma(t))a(\sigma(t))} \\ &\geq z(t) \frac{[a(t)y'(t)]\sigma'(t)}{y(\sigma(t))a(\sigma(t))} = z^2(t) \frac{\sigma'(t)}{a(\sigma(t))} \end{aligned} \quad (8)$$

由(6),(7)和(8)得

$$z'(t) \leq -cq(t) - \frac{\sigma'(t)}{a(\sigma(t))}z^2(t) \quad (9)$$

由此可得

$$cq(t) \leq -z'(t) - \frac{\sigma'(t)}{a(\sigma(t))}z^2(t) \quad (10)$$

于是, 对于任意的  $t > t_2 \geq t_1$ , 有

$$\begin{aligned} &\int_{t_2}^t cH(t, s)\rho(s)q(s)ds \\ &\leq -\int_{t_2}^t H(t, s)\rho(s)z'(s)ds - \int_{t_2}^t H(t, s)\rho(s) \frac{\sigma'(s)}{a(\sigma(s))}z^2(s)ds \\ &= H(t, t_2)\rho(t_2)z(t_2) + \int_{t_2}^t \frac{\partial(H(t, s)\rho(s))}{\partial s}z(s)ds - \int_{t_2}^t H(t, s)\rho(s) \frac{\sigma'(s)}{a(\sigma(s))}z^2(s)ds \\ &= H(t, t_2)\rho(t_2)z(t_2) - \int_{t_2}^t h(t, s)\sqrt{H(t, s)\rho(s)}z(s)ds - \int_{t_2}^t H(t, s)\rho(s) \frac{\sigma'(s)}{a(\sigma(s))}z^2(s)ds \\ &= H(t, t_2)\rho(t_2)z(t_2) - \int_{t_2}^t \left[ \sqrt{\frac{H(t, s)\rho(s)\sigma'(s)}{a(\sigma(s))}}z(s) + \frac{1}{2}h(t, s)\sqrt{\frac{a(\sigma(s))}{\sigma'(s)}} \right]^2 ds + \int_{t_2}^t \frac{a(\sigma(s))}{4\sigma'(s)}h^2(t, s)ds \\ &\leq H(t, t_2)\rho(t_2)z(t_2) + \int_{t_2}^t \frac{a(\sigma(s))}{4\sigma'(s)}h^2(t, s)ds \end{aligned}$$

进一步有

$$\int_{t_2}^t \left[ cH(t, s)\rho(s)q(s) - \frac{a(\sigma(s))}{4\sigma'(s)}h^2(t, s) \right] ds \leq H(t, t_2)\rho(t_2)z(t_2)$$

由  $\frac{\partial H(t, s)}{\partial s} \leq 0$ , 得  $H(t, t_2) \leq H(t, t_0)$  ( $t_2 \geq t_0$ ), 因此有

$$\int_{t_2}^t \left[ cH(t, s)\rho(s)q(s) - \frac{a(\sigma(s))}{4\sigma'(s)}h^2(t, s) \right] ds \leq H(t, t_0)\rho(t_2)z(t_2)$$

于是

$$\frac{1}{H(t, t_0)} \int_{t_2}^t \left[ cH(t, s)\rho(s)q(s) - \frac{a(\sigma(s))}{4\sigma'(s)}h^2(t, s) \right] ds \leq \rho(t_2)z(t_2)$$

进一步有

$$\begin{aligned} &\frac{1}{H(t, t_0)} \int_{t_0}^t \left[ cH(t, s)\rho(s)q(s) - \frac{a(\sigma(s))}{4\sigma'(s)}h^2(t, s) \right] ds \\ &= \frac{1}{H(t, t_0)} \int_{t_0}^{t_2} \left[ cH(t, s)\rho(s)q(s) - \frac{a(\sigma(s))}{4\sigma'(s)}h^2(t, s) \right] ds + \frac{1}{H(t, t_0)} \int_{t_2}^t \left[ cH(t, s)\rho(s)q(s) - \frac{a(\sigma(s))}{4\sigma'(s)}h^2(t, s) \right] ds \\ &\leq \int_{t_0}^{t_2} c \frac{H(t, s)}{H(t, t_0)} \rho(s)q(s)ds + \rho(t_2)z(t_2) \\ &\leq \int_{t_0}^{t_2} c\rho(s)q(s)ds + \rho(t_2)z(t_2) \end{aligned}$$

令  $t \rightarrow \infty$ , 并取上极限得

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left[ cH(t, s)\rho(s)q(s) - \frac{a(\sigma(s))}{4\sigma'(s)}h^2(t, s) \right] ds < \infty$$

此与(4)矛盾. 定理1证毕.

由定理1的证明过程易得

**推论 1** 若定理 1 的(4) 式被替换为

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t H(t, s) \rho(s) q(s) ds = \infty \quad (11)$$

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \frac{a(\sigma(s))}{\sigma'(s)} h^2(t, s) ds < \infty \quad (12)$$

则方程(1) 的所有解振动.

**定理 2** 假设条件(H<sub>1</sub>) – (H<sub>3</sub>) 成立, 如果存在常数  $m \geq 2$  和函数  $\rho(s) \in C'([t_0, \infty), (0, \infty))$ , 使得

$$\limsup_{t \rightarrow \infty} \frac{1}{t^m} \int_{t_0}^t (t-s)^m \rho(s) q(s) ds = \infty \quad (13)$$

$$\limsup_{t \rightarrow \infty} \frac{1}{t^m} \int_{t_0}^t \frac{a(\sigma(s))}{\sigma'(s) \rho(s)} (t-s)^{m-2} [m\rho(s) - (t-s)\rho'(s)]^2 ds < \infty \quad (14)$$

则方程(1) 的所有解振动.

**证** 不失一般性, 假设方程(1) 存在最终正解  $x(t)$ , 则由定理 1 的证明可知存在  $t_1 \geq t_0$ , 使得当  $t \geq t_1$  时, 有

$$cq(t) \leq -z'(t) - \frac{\sigma'(t)}{a(\sigma(t))} z^2(t)$$

于是, 对于任意的  $t \geq t_1$ , 有

$$\begin{aligned} & \int_{t_1}^t c(t-s)^m \rho(s) q(s) ds \\ & \leq - \int_{t_1}^t (t-s)^m \rho(s) z'(s) ds - \int_{t_1}^t (t-s)^m \rho(s) \frac{\sigma'(s)}{a(\sigma(s))} z^2(s) ds \\ & = (t-t_1)^m \rho(t_1) z(t_1) + \int_{t_1}^t [-m(t-s)^{m-1} \rho(s) + (t-s)^m \rho'(s)] z(s) ds - \int_{t_1}^t (t-s)^m \rho(s) \frac{\sigma'(s)}{a(\sigma(s))} z^2(s) ds \\ & = (t-t_1)^m \rho(t_1) z(t_1) - \int_{t_1}^t (t-s)^{m-1} [m\rho(s) - (t-s)\rho'(s)] z(s) ds - \int_{t_1}^t (t-s)^m \rho(s) \frac{\sigma'(s)}{a(\sigma(s))} z^2(s) ds \\ & = (t-t_1)^m \rho(t_1) z(t_1) - \int_{t_1}^t \left\{ \sqrt{\frac{\sigma'(s)}{a(\sigma(s))}} \rho(s) (t-s)^{\frac{m}{2}} z(s) + \frac{(t-s)^{\frac{m}{2}-1} [m\rho(s) - (t-s)\rho'(s)]}{2\sqrt{\frac{\sigma'(s)}{a(\sigma(s))} \rho(s)}} \right\}^2 ds + \\ & \quad \int_{t_1}^t \frac{a(\sigma(s))(t-s)^{m-2}}{4\sigma'(s)\rho(s)} [m\rho(s) - (t-s)\rho'(s)]^2 ds \\ & \leq (t-t_1)^m \rho(t_1) z(t_1) + \int_{t_1}^t \frac{a(\sigma(s))(t-s)^{m-2}}{4\sigma'(s)\rho(s)} [m\rho(s) - (t-s)\rho'(s)]^2 ds \end{aligned} \quad (15)$$

进一步有

$$\begin{aligned} & \frac{1}{t^m} \int_{t_1}^t c(t-s)^m \rho(s) q(s) ds \\ & \leq \frac{1}{t^m} (t-t_1)^m \rho(t_1) z(t_1) + \frac{1}{t^m} \int_{t_1}^t \frac{a(\sigma(s))(t-s)^{m-2}}{4\sigma'(s)\rho(s)} [m\rho(s) - (t-s)\rho'(s)]^2 ds \end{aligned}$$

于是, 有

$$\begin{aligned} & \frac{1}{t^m} \int_{t_0}^t c(t-s)^m \rho(s) q(s) ds \\ & = \frac{1}{t^m} \int_{t_0}^{t_1} c(t-s)^m \rho(s) q(s) ds + \frac{1}{t^m} \int_{t_1}^t c(t-s)^m \rho(s) q(s) ds \\ & \leq \frac{1}{t^m} \int_{t_0}^{t_1} c(t-s)^m \rho(s) q(s) ds + \frac{1}{t^m} (t-t_1)^m \rho(t_1) z(t_1) + \frac{1}{t^m} \int_{t_1}^t \frac{a(\sigma(s))(t-s)^{m-2}}{4\sigma'(s)\rho(s)} [m\rho(s) - (t-s)\rho'(s)]^2 ds \\ & \leq \int_{t_0}^{t_1} c \left(1 - \frac{s}{t}\right)^m \rho(s) q(s) ds + \left(1 - \frac{t_1}{t}\right)^m \rho(t_1) z(t_1) + \frac{1}{t^m} \int_{t_0}^t \frac{a(\sigma(s))(t-s)^{m-2}}{4\sigma'(s)\rho(s)} [m\rho(s) - (t-s)\rho'(s)]^2 ds \end{aligned}$$

令  $t \rightarrow \infty$ , 并取上极限得

$$\limsup_{t \rightarrow \infty} \frac{1}{t^m} \int_{t_0}^t c(t-s)^m \rho(s) q(s) ds$$

$$\leq \int_{t_0}^{t_1} c \rho(s) q(s) ds + \rho(t_1) z(t_1) + \limsup_{t \rightarrow \infty} \frac{1}{t^m} \int_{t_0}^t \frac{a(\sigma(s))(t-s)^{m-2}}{4\sigma'(s)\rho(s)} [m\rho(s) - (t-s)\rho'(s)]^2 ds < \infty$$

此与(13)矛盾. 定理 2 证毕.

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## Oscillation Criteria for Nonlinear Neutral Differential Equations

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**Abstract:** A class of second order nonlinear neutral differential equations is discussed. Oscillatory criteria of the equations are obtained. The corresponding results in known literature are improved and extended.

**Key words:** nonlinear; neutral differential equation; oscillation

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