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Triple Positive Solutions of a Kind of Nonlinear Boundary Value Problem①

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Abstract: In this paper, the authors consider the nonlinear boundary value problem (BVP):

$$\begin{cases} -y^{(6)}(t) = f(y(t), -y''(t), y^{(4)}(t)) & t \in [0, 1] \\ y(0) = y'(1) = 0 \\ y''(0) = y''(1) = 0 \\ y^{(4)}(0) = y^{(4)}(1) = 0 \end{cases}$$

where, $f>0$. The boundary conditions are different from the Lidstone boundary conditions. By using the Leggett-Williams Fixed Point Theroem and inequalities involving an associated green s function, a sufficient condition for the existence of triple positive concave solutions of BVP is obtained.

Key words: boundary value problem; triple positive solutions; fixed points; Green s function

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Consider the nonlinear boundary value problem(BVP)

$$\begin{cases} -y^{(6)}(t) = f(y(t), -y''(t), y^{(4)}(t)) & t \in [0, 1] \\ y(0) = y'(1) = 0 \\ y''(0) = y''(1) = 0 \\ y^{(4)}(0) = y^{(4)}(1) = 0 \end{cases} \tag{1}$$

Where $f: [0, \infty) \times [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$ is continuous.

The above BVP arose from the Lidstone boundary value problem[1]. We have mainly changed the boundary conditions. Lidstone boundary value problems for ordinary differential equations have enjoyed much attention lately[1-2]. The proof of the existence of triple positive solutions in (1) is the application of a multiple fixed point theorem by [3]. Motivated by the method of [1], we consider the existence of at least three positive solutions to (1) by using the Leggett-Williams Fixed Point Theorem[3].

By a solution of (1), we mean a c^6 function on [0, 1] satisfying (1). We will impose growth conditions on f and make use of inequalities involving an associated Green's function to consider the existence of at least three positive solution to (1).

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The main results will hinge on an application of the Leggett-williams fixed point theorem. We first introduce the necessary definitions and theorem from cone theory in Banach spaces.

Let B a Banach space over R. A nonempty, closed set $P \subset B$ is a cone

Definition 1 The map α is a nonnegative continuous concave functional on P provided $\alpha: P \longrightarrow [0, \infty)$ is continuous and $\alpha(tx+(1-t)y) \geqslant t\alpha(x)+(1-t)\alpha(y)$, for all $x, y \in P$ and $t \in [0, 1]$.

Definition 2 Let $0<a<b$, $r>0$ and α be a nonnegative continuous concave functional on the cone P. Define the convex sets P_r and $P(\alpha, a, b)$ by

$$P_r = \{y \in P \mid \| y \| < r\} \qquad P(\alpha, a, b) = \{y \in P \mid a \leqslant \alpha(y), \ \| y \| \leqslant b\}$$

Leggett-Williams Fixed Point Theorem[3]. Let $A: P_c \longrightarrow P_c$ be a completely continuous operator and let α be a nonnegative continuous concave functional on P such that $\alpha(y) \leqslant \| y \|$ for all $y \in P_c$. Suppose there exist $0<a<b<d \leqslant c$ such that

(C_1) $\{y \in P(\alpha, b, d) \mid \alpha(y) > b\} \neq \varnothing$. And $\alpha(Ay) > b$, for $y \in P(\alpha, b, d)$;

(C_2) $\| Ay \| < a$. For $\| y \| \leqslant a$;

(C_3) $\alpha(Ay) > b$. For $y \in P(\alpha, b, c)$ with $\| Ay \| > d$. Then A has at least three fixed points y_1, y_2, y_3 such that $\| y_1 \| < a$, $b < \alpha(y_2)$ and $\| y_3 \| > a$ with $\alpha(y_3) < b$.

We Define operator K, H, HK : $C[0, 1] \longrightarrow C[0, 1]$, by

$$\begin{aligned} Kv(t) &= \int_0^1 G(t, s)v(s)\mathrm{d}s && t \in [0, 1], \ \forall v \in C[0, 1] \\ Hw(t) &= \int_0^1 E(t, s)v(s)\mathrm{d}s && t \in [0, 1], \ \forall v \in C[0, 1] \\ HKv(t) &= \int_0^1 G(t, s)Hv(s)\mathrm{d}s && t \in [0, 1], \ \forall v \in C[0, 1] \end{aligned} \tag{2}$$

Where $G(t, s)$, $E(t, s)$ are the Green's function for $y''(t)=0$, $t \in [0, 1]$, $y(0)=y(1)=0$, $y''(t)=0$, $t \in [0, 1]$, $y(0)=y'(1)=0$ respectly, ie

$$G(t, s) = \begin{cases} t(1-s) & 0 \leqslant t \leqslant s \leqslant 1 \\ s(1-t) & 0 \leqslant s \leqslant t \leqslant 1 \end{cases}$$

$$E(t, s) = \min(t, s) = \begin{cases} t & 0 \leqslant t \leqslant s \leqslant 1 \\ s & 0 \leqslant s \leqslant t \leqslant 1 \end{cases}$$

In the following lemma, we get a second order BVP that is equivalent to (1).

Lemma 1 Let $f: R^1 \times R^1 \times R^1 \longrightarrow R^1$, then (1) has a solution u if and only if, the BVP

$$\begin{cases} -v''(t) = f(HKv(t), Kv(t), v(t)) & 0 \leqslant t \leqslant 1 \\ v(0) = v(1) = 0 \end{cases} \tag{3}$$

has a solution v and $v(t) = u^{(4)} = \int_0^1 G(t, s) f(HKv(s), Kv(s), v(s))\,\mathrm{d}s$.

Proof Let u be a solution of (1), and let $v(t) = u^{(4)}(t)$, then (1) is equivalent to

$$\begin{cases} -v'' = f(u, -u'', v) & 0 \leqslant t \leqslant 1 \\ v(0) = v(1) = 0 \end{cases} \tag{4}$$

$$\begin{cases} u^{(4)}(t) = v(t) & t \in [0, 1] \\ u(0) = u'(1) = 0 \\ u''(0) = u''(1) = 0 \end{cases} \tag{5}$$

let $w(t) = -u''(t)$, then (5) can be changed into

$$\begin{cases} -u'' = w(t) & 0 \leqslant t \leqslant 1 \\ u(0) = u'(1) = 0 \end{cases}$$

$$\begin{cases} -w'' = v(t) & 0 \leqslant t \leqslant 1 \\ w(0) = w(1) = 0 \end{cases}$$

By using (2), we see $w(t) = \int_0^1 G(t, s)v(s)\mathrm{d}s = Kv(s)$, and $u(t) = \int_0^1 E(t, s)w(s)\mathrm{d}s = Hw(s)$. Due to $-u''(t) = w(t) = Kv(t)$, then $u(t) = Hw(t) = HKv(t)$. So

$$v(t) = \int_0^1 G(t, s)f(HKv(s), Kv(s), v(s))\mathrm{d}s \qquad v(0) = v(1) = 0$$

If y is a solution of (1), then $v(t) = y^{(4)}(t)$ is a solution of the second order BVP, conversely, if v is a solution of the second order BVP, then $y = HKv$ is a solution of (1). The proof of Lemma is complete.

Define $A: C[0, 1] \longrightarrow C[0, 1]$ by $Av(t) = \int_0^1 G(t, s)f(HKv(s), Kv(s), v(s))\mathrm{d}s$.

It follows that there exists a solution of (1) if and only if, there exists a continuous fixed point of A. Moreover the relationship between a solution, y, of (1), and a fixed point, v, of A is given by $y(t) = HKv(t)$.

Next, we restrict our analysis to show that A generates at least three distinct solutions, v_1, v_2, v_3, such that $v_i > 0$, $i = 1, 2, 3$.

We will utilize the following inequalities involving the aforementioned Green's function

$$G(t, s) \leqslant s(1-s) \qquad 0 < t, s < 1$$

$$G(t, s) \geqslant s(1-s)/4 \qquad 1/4 \leqslant t \leqslant 3/4, \ 0 \leqslant s \leqslant 1$$

$$\max_{0 \leqslant t \leqslant 1} \int_0^1 G(t, s)\mathrm{d}s = \frac{1}{8} \tag{7}$$

$$\max_{0 \leqslant t \leqslant 1} \int_0^1 E(t, s)\mathrm{d}s = \frac{1}{2} \tag{8}$$

$$\min_{\frac{1}{4} \leqslant t \leqslant \frac{3}{4}} \int_0^1 E(t, s)\mathrm{d}s = \frac{1}{8} \tag{9}$$

$$\min_{\frac{1}{4} \leqslant t \leqslant \frac{3}{4}} \int_0^1 G(t, s)\mathrm{d}s = \frac{1}{16} \tag{10}$$

We now present the result of the paper which establishes the existence of three positive concave solutions of the BVP (1).

Theorem 1　Let $0 < 4a < b \leqslant c$ be given and suppose that f satisfies

(i) $f: [0, a/16] \times [0, a/8] \times [0, a] \longrightarrow [0, 8a)$;

(ii) $f: [b/128, b/4] \times [b/16, b/2] \times [b, 4b] \longrightarrow [16b, \infty)$;

(iii) $f: [0, c/16] \times [0, c/8] \times [0, c] \longrightarrow [0, 8c]$.

Then the BVP (1) has three positive solutions, y_1, y_2, y_3, such that

$$\| y_1^{(4)} \| < a \qquad b < \min_{\frac{1}{4} \leqslant t \leqslant \frac{3}{4}} y_2^{(4)}(t)$$

$$\| y_3^{(4)} \| > a \qquad b > \min_{\frac{1}{4} \leqslant t \leqslant \frac{3}{4}} y_3^{(4)}(t)$$

Proof　Let B denote the Banach space $C[0, 1]$ with the maximum norm $\| v \| = \max_{0 \leqslant t \leqslant 1} | v(t) |$ and define the cone $P \subset B$ by

$$p = \{ v \in B: v(t) \geqslant 0, \text{ for } 0 \leqslant t \leqslant 1, \ v \text{ is concave, and } \min_{\frac{1}{4} \leqslant t \leqslant \frac{3}{4}} v(t) \geqslant \frac{1}{4} \| v \| \}$$

Let $\alpha: P \longrightarrow [0, \infty)$ be the nonnegative continuous concave functional

$$\alpha(v) = \min_{\frac{1}{4} \leqslant t \leqslant \frac{3}{4}} v(t)$$

for $v \in P$ and let $A : B \longrightarrow B$ be the operator

$$Av(t) = \int_0^1 G(t, s) f(HKv(s), Kv(s), v(s)) \mathrm{d}s$$

We will first verify that $A : P \longrightarrow P$.

Let $v \in P$. From the properties of $G(t, s)$ and $E(t, s)$, and the assumption on f, it follows that $Av(t) \geqslant 0$, and $(Av)''(t) = -f(HKv(t), Kv(t), v(t)) \leqslant 0$, $t \in [0, 1]$, so Av is concave, Moreover

$$\begin{aligned} Av(t) &\geqslant \frac{1}{4}\int_0^1 s(1-s) f(HKv(s), Kv(s), v(s)) \mathrm{d}s \\ &\geqslant \frac{1}{4}\int_0^1 G(\tau, s) f(HKv(s), Kv(s), v(s)) ds = \frac{1}{4} Av(\tau) \qquad \tau \in [0, 1], s \in [0, 1] \end{aligned}$$

In particular, $\min_{\frac{1}{4} \leqslant t \leqslant \frac{3}{4}} Av(t) \geqslant \frac{1}{4} \| Av \|$.

It is obvious to see that A: $P \longrightarrow P$ and A is completely continuous.

Let $v \in P_c$, then $\| v \| \leqslant c$ and

$$\| Kv \| = \| \int_0^1 G(t, s) v(s) \mathrm{d}s \| \leqslant c\int_0^1 G(t, s) \mathrm{d}s \leqslant \frac{c}{8}$$

$$\| Hv \| = \| \int_0^1 E(t, s) v(s) \mathrm{d}s \| \leqslant c\int_0^1 E(t, s) \mathrm{d}s \leqslant \frac{c}{2}$$

$$\| HKv \| = \| \int_0^1 G(t, s) Hv(s) \mathrm{d}s \| \leqslant \frac{c}{16}$$

Thus assumption (iii) implies $f(HKv(s), Kv(s), v(s)) \leqslant 8c$, for $0 \leqslant s \leqslant 1$. From this and (7) we obtain:

$$\begin{aligned} \| Av \| &= \max_{0 \leqslant t \leqslant 1} | Av(t) | \\ &= \max_{0 \leqslant t \leqslant 1} \int_0^1 G(t, s) f(HKv(s), Kv(s), v(s)) \mathrm{d}s \\ &\leqslant 8c \max_{0 \leqslant t \leqslant 1} \int_0^1 G(t, s) \mathrm{d}s \leqslant c \end{aligned}$$

So A: $P_c \longrightarrow P_c$.

Assumption (i) implies that condition (C_2) of the Leggett-Williams Theorem is satisfied.

We now show that condition (C_1) is satisfied. Note that for $0 \leqslant t \leqslant 1$, $v(t) = 4b \in P(\alpha, b, 4b)$ and $a(v) = 4b > b$. Thus $\{v \in P(\alpha, b, 4b) \mid \alpha(v) > b\} \neq \varnothing$. Also, if $v \in P(\alpha, b, 4b)$, then $\alpha(v) = v(1/4) \geqslant b$, and $b \leqslant v(s) \leqslant 4b$, for each $1/4 \leqslant s \leqslant 3/4$. Apply (8)(9) and we see that $v(s) \geqslant b$, $1/4 \leqslant s \leqslant 3/4$, implies $Kv(s) \geqslant b/16$ and $HKv(s) \geqslant b/128$, $1/4 \leqslant s \leqslant 3/4$. Then $b \leqslant v(s) \leqslant 4b$, $b/16 \leqslant Kv(s) \leqslant 4b/8$, $b/128 \leqslant HKv(s) \leqslant b/4$.

Assumption (ii) now applies $f(HKv(s), Kv(s), v(s)) \geqslant 16b$, $1/4 \leqslant s \leqslant 3/4$.

This together with (10) yields

$$\begin{aligned} \alpha(Av) = \min_{\frac{1}{4} \leqslant t \leqslant \frac{3}{4}} Av(t) &= \min_{\frac{1}{4} \leqslant t \leqslant \frac{3}{4}} \int_0^1 G(t, s) f(HKv(s), Kv(s), v(s)) \mathrm{d}s \\ &> \min_{\frac{1}{4} \leqslant t \leqslant \frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G(t, s) f(HKv(s), Kv(s), v(s)) \mathrm{d}s \\ &> 16b \min_{\frac{1}{4} \leqslant t \leqslant \frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s) \mathrm{d}s = b \end{aligned}$$

Therefore，condition (C_1) is satisfied.

Finally，we show that condition (C_3) is also satisfied. That is，we show that if $v \in P(\alpha, b, c)$ and $\| Av \| > d = 4b$. Then $\alpha(Av) > b$. This follows since $A: P \longrightarrow P$. In particular，since Av is concave

$$\min_{\frac{1}{4} \leqslant t \leqslant \frac{3}{4}} Av(t) \geqslant \frac{1}{4} \| Av \|$$

That is $\alpha(Av) > d/4 = b$. therefore，(C_3) is satisfied . An application of the Leggett-Williams Fixed Point Theorem yields the result.

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一类非线性边值问题的三重正解

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摘要：研究边值问题：

$$\begin{cases} -y^{(6)}(t) = f(y(t), -y''(t), y^{(4)}(t)) & 0 \leqslant t \leqslant 1 \\ y(0) = y'(1) = 0 \\ y''(0) = y''(1) = 0 \\ y^{(4)}(0) = y^{(4)}(1) = 0 \end{cases}$$

其中 $f \geqslant 0$ ，其边值条件不同于 Lidstone 边值条件，应用 Leggett-williams 不动点定理和格林函数得到边值问题存在三重正解的充分条件.

关键词：边值问题；三重正解；不动点；格林函数

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