文章编号：1000－5471(2007)01－0011－04

# General Convergence Analysis for Generalized $n$-Step Projection Methods and Applications①

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**Abstract**：In this paper，the authors introduce and study a new system of nonlinear variational inequalities and give out a new $n$-step iterative algorithm. The convergence of a new $n$-step iterative algorithm for this system of nonlinear variational inequalities is proved.

**Key words**：generalized $n$-step projection methods；nonlinear variational inequality；convergence

**CLC number**：O177.91　　**Document code**：A

Projection methods have played a significant role in the numerical resolution of variational inequalities based in their convergence analyses. However，the convergence analysis does require some sort of strong monotonicity besides the Lipschitz continuity. There have been some recent developments where convergence analysis for projection/projection type methods under somewhat weaker conditions such as coercivity[1,2] and partial relaxed monotonicity[3] is achieved. [4] introduced a two-step model for nonlinear variational inequalities and discussed the approximation solvability of this model based on convergence analysis of two-step projection method in Hilbert space setting. More recently，[5] obtained the new result of general two-step model for projection methods. The general two-step models for nonlinear variational inequality problems and their corresponding solvability. And in [6]，had shown the NNVI(1.1)exists only one solution $x_i^* \in K_i$，$i=1,2,\cdots,n$ Here in this paper，a system of n-nonlinear variational inequality in [6] is introduced. And give the general $n$-step projection methods for this new model. At last，the convergence analysis of this system is shown.

Let $H$ be a real Hilbert space with the inner product$\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$，Let $T_i$：$H \longrightarrow H$，$i=1$，$2,\cdots,n$ are nonlinear mappings on $K_i$，$i=1,2,\cdots,n$；$\phi_i$：$H \longrightarrow R \cup \{+\infty\}$ $(i=1,2,\cdots,n)$ is a solid convex lower semi-continuous functions；$K_i$，$i=1,2,\cdots,n$ be a nonempty closed convex subset of $H$. We consider a system of generalized $n$-nonlinear variational inequality (abbreviated NNVI) problems as following：

Determine elements $x_i^* \in K_i$，$i=1,2,\cdots,n$ such that：

$$\langle \rho_i T_n(x_{i+1}^*) + x_i^* - x_{i+1}^*,\ x - x_i^* \rangle \geqslant \rho_i \phi_i(x_i^*) - \rho_i \phi_i(x) \qquad \forall x \in K_i,\ \rho_i > 0 \tag{1}$$

there into $i=1,2,\cdots,n,n+1 \doteq 1$.

The NNVI problems are equivalent to the following projection formulas：

$$x_i^* = J_{\partial\phi_i}^{\rho}[x_{i+1}^* - \rho_i T_i(x_{i+1}^*)] \qquad i=1,2,\cdots,n,n+1 \doteq 1 \tag{2}$$

where $J_{\partial\phi_i}^{\rho}$，$i=1,2,\cdots,n$ are the resolvent operators of $\partial\phi_i$，that is $J_{\partial\phi_i}^{\rho} = (I+\partial\phi_i)^{-1}$，$i=1,2,\cdots,n$.

**Remark 1**　If we let $\phi_i = \delta_{K_i}$ (the directive function of nonempty closed convex sets $K_i \subset H$)，then this generalized $n$-nonlinear variational inequality (abbreviated NNVI) problems are equal to the following

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①　收稿日期：2005－12－06

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problems.

Determine elements $x_i^* \in K_i$, $i = 1,2,\cdots,n$ such that

$$\langle \rho_i T_i(x_{i+1}^*) + x_i^* - x_{i+1}^*,\ x - x_i^* \rangle \geqslant 0 \qquad \forall x \in K_i,\ \rho_i > 0 \tag{3}$$

there into $i = 1,2,\cdots,n,n+1 \doteq 1$.

The NNVI problems are equivalent to the following projection formulas

$$x_i^* = P_{k_i}[x_{i+1}^* - \rho_i T_i(x_{i+1}^*)] \qquad i = 1,2,\cdots,n,n+1 \doteq 1 \tag{4}$$

where $P_{K_i,\ i=1,2,\cdots,n}$ are the projection of $H$ onto $K_i$, $i = 1,2,\cdots,n$.

**Remark 2** Let $n = 2$, $T_i = T$, $K_i = K$, $i = 1,2$. then the NNVI is reduced to:

$$\langle \rho_1 T(x_2^*) + x_1^* - x_2^*,\ x - x_1^* \rangle \geqslant 0 \qquad \forall x \in K,\ \rho_1 > 0$$
$$\langle \rho_2 T_2(x_1^*) + x_2^* - x_1^*,\ x - x_2^* \rangle \geqslant 0 \qquad \forall x \in K,\ \rho_2 > 0$$

that is the NNVI is reduced to the SNVI in [5]. And if we let $n = 2$, $T_i = T$, $K_i = K$, $i = 1,2$. Let $K$ be a closed convex cone of $H$, then the NNVI problem is equivalent to a system of nonlinear complementarities:

find the elements $x_1^*, x_2^* \in K$, such that $T(x_1^*) \in K$, and

$$\langle \rho_1 T(x_2^*) + x_1^* - x_2^*,\ x_1^* \rangle \geqslant 0 \qquad \forall x \in K,\ \rho_1 > 0$$
$$\langle \rho_2 T_2(x_1^*) + x_2^* - x_1^*,\ x_2^* \rangle \geqslant 0 \qquad \forall x \in K,\ \rho_2 > 0$$

**Definition 1** A mapping $T_i: H \longrightarrow H$, $i = 1,2,\cdots,n$ is called $\gamma_i$, strongly monotonic, if for each $x$, $y \in H$, we have

$$\langle T_i(x) - T_i(y),\ x - y \rangle \geqslant \gamma_i \| x - y \|^2 \qquad \forall \gamma_i > 0,\ i = 1,2,\cdots,n \tag{5}$$

# 1 Projection Methods

In this section, we give a $n$-step projection method for the NNVI problems as follows.

**Algorithm 1** For arbitrarily chosen initial points $x_{1,0}, x_{2,0}, \cdots, x_{n,0} \in H$, compute sequences $\{x_{1,k+1}\}$, $\{x_{2,k}\}, \cdots, \{x_{n,k}\}$ such that

$$x_{1,k+1} = (1 - \alpha_{1,k} - r_{1,k})x_{1,k} + \alpha_{1,k} J_{\partial\phi_1}^{\rho}[x_{2,k} - \rho_1 T_1(x_{2,k})] + r_{1,k}u_{1,k}$$
$$x_{i,k} = (1 - \alpha_{i,k} - r_{i,k})x_{1,k} + \alpha_{i,k} J_{\partial\phi_i}^{\rho}[x_{i+1,k} - \rho_i T_i(x_{i+1,k})] + r_{i,k}u_{i,k}$$

there into $i = 2,\cdots,n,n+1 \doteq 1$. For $\rho_i > 0$, $i = 1,2,\cdots,n$ are constants, $\{\alpha_{i,k}\}$, $\{r_{i,k}\} \subset [0,\ 1]$, $\{u_{i,k}\}$($i = 1,2,\cdots,n$) are the sequences in $K_i$, $i = 1,2,\cdots,n$; and $0 \leqslant \alpha_{i,k} + r_{i,k} \leqslant 1$, $i = 1,2,\cdots,n$ where $T_i: K_i \longrightarrow H$, $i = 1,2,\cdots,n$ are nonlinear mappings on $K_i$, $i = 1,2,\cdots,n$; $\phi_i: H \longrightarrow R \cup \{+\infty\}$($i = 1,2,\cdots,n$) is a solid convex lower semi-continuous functions.

**Algorithm 2** Let $\phi_i = \delta_{K_i}$ (the directive function of nonempty closed convex sets $K_i \in H$), then we may get another algorithm(Algorithm 2) by put $J_{\partial\phi_i}^{\rho}[x_{i+1,k} - \rho_i T_i(x_{i+1,k})]$ into$P_{K_i}[x_{i+1,k} - \rho_i T_i(x_{i+1,k})]$, $i = 1,2,\cdots,n,n+1 \doteq 1$.

Where $T_i: K_i \longrightarrow H$, $i = 1,2,\cdots,n$ are nonlinear mappings on $K_i$, $i = 1,2,\cdots,n$.

**Remark 3** By letting $n = 2$, $T_i = T$, $K_i = K$, $r_i = 0$, $i = 1,2,\cdots,n$, then the Algorithm 1 is reduced to the Algorithm 2.1 in [5]. So the algorithm in this paper is absolutely new.

# 2 Applications

We now present, based on Algorithm 2.1, the approximation-solvability of the NNVI problems involving $\gamma_i$-strongly monotonic and $\mu_i$-Lipschitz continuous mappings in a Hilbert space setting, for $i = 1,2,\cdots,n$.

**Theorem 1** Let $H$ be a real Hilbert space and $K_i$, $i = 1,2,\cdots,n$, be nonempty closed convex subsets of $H$. Let $T_i: H \longrightarrow H$ be $\gamma_i$-strongly monotonic and $\mu_i$-Lipschitz continuous mappings, for $i = 1,2,\cdots,n$. Suppose that $x_i^* \in K_i$, $i = 1,2,\cdots,n$ form a solution to the NNVI problems, the sequences $\{x_{1,k+1}\}$, $\{x_{2,k}\}, \cdots, \{x_{n,k}\}$ are generated by Algorithm 1 and $\{\alpha_{i,k}\}, \{r_{i,k}\} \subset [0,\ 1]$, $i = 1,2,\cdots,n$, and if the following holds true

(i) $\alpha_{i,k} \to 1$, $r_{i,k} \to 0$, $\sum_{k=0}^{\infty} r_{i,k} < \infty$, $\sum_{k=0}^{\infty} \alpha_{i,k} = \infty$ $(k \to \infty)$.

(ii) $0 < \rho_i < 2\gamma_i / \mu_i^2$, $i = 1,2,\cdots,n$.

(iii) $\mu_{i,k}$ is bounded sequences in $K_i$, $i = 1,2,\cdots,n$.

Then $x_{1,k+1} \to x_1^*$, $x_{2,k} \to x_2^*$, $\cdots$, $x_{n,k} \to x_n^*$ $(k \to \infty)$.

**Proof**　Since $x_i^* \in K_i$, $i = 1,2,\cdots,n$ form a solution to the NNVI problems, it follows that

$$x_i^* = J^{\rho_i}_{\partial\phi_i}[x_{i+1}^* - \rho_i T_i(x_{i+1}^*)] \qquad i = 1,2,\cdots,n,n+1 \doteq 1$$

Let:

$$M = \max\{\sup_{k\geqslant 0}\| u_{1,k} - x_1^* \|, \sup_{k\geqslant 0}\| u_{2,k} - x_2^* \|, \cdots, \sup_{k\geqslant 0}\| u_{n,k} - x_n^* \|, \| x^*_{1-x_2^*} \|, \cdots, \| x^*_{1-x_n^*} \|\}$$

Applying Algorithm1, we have

$$\begin{aligned}\| x_{1,k+1} - x_1^* \| &= \| (1-\alpha_{1,k}-r_{1,k})(x_{1,k}-x_1^*) + \alpha_{1,k}\{J^{\rho_1}_{\partial\phi_1}[x_{2,k}-\rho_1 T_1(x_{2,k})] - J^{\rho_1}_{\partial\phi_1}[x_2^* - \rho_1 T_1(x_2^*)]\} + r_{1,k}(u_{1,k}-x_1^*) \| \\ &\leqslant (1-\alpha_{1,k}-r_{1,k})\| x_{1,k}-x_1^* \| + \alpha_{1,k}\| x_{2,k}-x_2^* - \rho_1[T_1(x_{2,k}) - T_1(x_2^*)] \| + r_{1,k}\| u_{1,k} - x_1^* \| \\ &\leqslant (1-\alpha_{1,k})\| x_{1,k}-x_1^* \| + \alpha_{1,k}\| x_{2,k}-x_2^* - \rho_1[T_1(x_{2,k}) - T_1(x_2^*)] \| + Mr_{1,k} \end{aligned} \tag{6}$$

Since $T_1$ is $\gamma_1$-strongly monotonic and $\mu_1$-Lipschitz continuous, we have

$$\begin{aligned}&\| x_{2,k} - x_2^* - \rho_1[T_1(x_{2,k}) - T_1(x_2^*)] \|^2 \\ &= \| x_{2,k} - x_2^* \|^2 - 2\rho_1\langle T_1(x_{2,k}) - T_1(x_2^*), x_{2,k} - x_2^* \rangle + \rho_1^2 \| T_1(x_{2,k}) - T_1(x_2^*) \|^2 \\ &\leqslant \| x_{2,k} - x_2^* \|^2 - 2\rho_1\gamma_1 \| x_{2,k} - x_2^* \|^2 + \rho_1^2\mu_1^2 \| x_{2,k} - x_2^* \|^2 \\ &= (1 - 2\rho_1\gamma_1 + \rho_1^2\mu_1^2)\| x_{2,k} - x_2^* \|^2 \\ &= \theta_1^2 \| x_{2,k} - x_2^* \|^2 \end{aligned} \tag{7}$$

Where ${\theta_1}^2 = 1 - 2\rho_1\gamma_1 + \rho_1^2\mu_1^2 \in (0, 1)$.

As a result, in light of (6), we have

$$\| x_{1,k+1} - x_1^* \| \leqslant (1-\alpha_{1,k})\| x_{1,k} - x_1^* \| + \alpha_{1,k}\theta_1 \| x_{2,k} - x_2^* \| + r_{1,k}M \tag{8}$$

Similarly, we have

$$\begin{aligned}\| x_{2,k} - x_2^* \| &= \| (1-\alpha_{2,k}-r_{2,k})(x_{1,k}-x_2^*) + \alpha_{2,k}\{J^{\rho_2}_{\partial\phi_2}[x_{3,k}-\rho_2 T_2(x_{3,k})] - J^{\rho_2}_{\partial\phi_2}[x_3^* - \rho_2 T_2(x_3^*)]\} + r_{2,k}(u_{2,k}-x_2^*) \| \\ &\leqslant (1-\alpha_{2,k}-r_{2,k})\| x_{2,k}-x_2^* \| + \alpha_{2,k}\| x_{3,k}-x_3^* - \rho_2[T_2(x_{3,k}) - T_2(x_3^*)] \| + r_{2,k}\| u_{2,k} - x_2^* \| \\ &\leqslant (1-\alpha_{2,k})\| x_{2,k}-x_2^* \| + \alpha_{2,k}\| x_{3,k}-x_3^* - \rho_2[T_2(x_{3,k}) - T_2(x_3^*)] \| + r_{2,k}M \\ &\leqslant (1-\alpha_{2,k})\| x_{2,k}-x_2^* \| + \alpha_{2,k}\| x_{3,k}-x_3^* - \rho_2[T_2(x_{3,k}) - T_2(x_3^*)] \| + (1-\alpha_{2,k}+r_{2,k})M \end{aligned} \tag{9}$$

Where ${\theta_2}^2 = 1 - 2\rho_2\gamma_2 + \rho_2^2\mu_2^2 \in (0, 1)$.

And by the same way, we have

$$\begin{aligned}\| x_{n,k} - x_n^* \| &= \| (1-\alpha_{n,k}-r_{n,k})(x_{1,k}-x_n^*) + \alpha_{n,k}\{J^{\rho_n}_{\partial\phi_n}[x_{1,k}-\rho_n T_n(x_{1,k})] - J^{\rho_n}_{\partial\phi_n}[x_1^* - \rho_n T_n(x_1^*)]\} + r_{n,k}(u_{n,k}-x_n^*) \| \\ &\leqslant (1-\alpha_{n,k}-r_{n,k})\| x_{1,k}-x_n^* \| + \alpha_{n,k}\| x_{1,k}-x_1^* - \rho_n[T_n(x_{1,k}) - T_n(x_1^*)] \| + r_{n,k}\| u_{n,k} - x_n^* \| \\ &\leqslant (1-\alpha_{n,k})\| x_{1,k}-x_n^* \| + \alpha_{n,k}\theta_n \| x_{1,k} - x_1^* \| + r_{n,k}M \\ &\leqslant (1-\alpha_{n,k})\| x_{1,k}-x_1^* \| + (1-\alpha_{n,k})\| x_1^* - x_n^* \| + \alpha_{n,k}\| x_{1,k} - x_1^* \| + r_{n,k}M \\ &\leqslant \| x_{1,k}-x_1^* \| + (1-\alpha_{n,k})\| x_1^* - x_n^* \| + r_{n,k}M \\ &\leqslant \| x_{1,k}-x_1^* \| + (1-\alpha_{n,k}+r_{n,k})M \end{aligned} \tag{10}$$

Where ${\theta_n}^2 = 1 - 2\rho_n\gamma_n + \rho_n^2\mu_n^2 \in (0, 1)$.

It follows from (8),(9) and (10) that

$$\begin{aligned}&\| x_{1,k+1} - x_1^* \| \leqslant (1-\alpha_{1,k})\| x_{1,k} - x_1^* \| + \alpha_{1,k}\theta_1 \| x_{2,k} - x_2^* \| + r_{1,k}M \\ &\leqslant (1-\alpha_{1,k})\| x_{1,k} - x_1^* \| + \alpha_{1,k}\theta_1[(1-\alpha_{2,k})\| x_{1,k} - x_1^* \| + \alpha_{2,k}\theta_2 \| x_{3,k} - x_3^* \| + (1-\alpha_{2,k}+r_{2,k})M] + r_{1,k}M \\ &\leqslant [(1-\alpha_{1,k}) + \alpha_{1,k}\theta_1(1-\alpha_{2,k})]\| x_{1,k} - x_1^* \| + \alpha_{1,k}\alpha_{2,k}\theta_1\theta_2 \| x_{3,k} - x_3^* \| + \alpha_{1,k}\theta_1(1-\alpha_{2,k}+r_{2,k})M + r_{1,k}M \\ &\quad \vdots \\ &\leqslant [(1-\alpha_{1,k}) + \alpha_{1,k}\theta_1(1-\alpha_{2,k}) + \cdots + \alpha_{1,k}\alpha_{2,k}\alpha_{n-1,k}\theta_1\theta_2\cdots\theta_{n-1}(1-\alpha_{n,k})]\| x_{1,k} - x_1^* \| + \\ &\quad [r_{1,k} + \alpha_{1,k}\theta_1(1-\alpha_{2,k}+r_{2,k}) + \cdots + \alpha_{1,k}\alpha_{2,k}\alpha_{n-1,k}\theta_1\theta_2\cdots\theta_{n-1}(1-\alpha_{n,k}+r_{n,k})]M \end{aligned}$$

By taking　$a_k = \| x_{1,k} - x_1^* \|$

$$t_k = \alpha_{1,k}\{1-[\theta_1(1-\alpha_{2,k})+\cdots+\alpha_{2,k},\ \cdots,\ \alpha_{n-1,k}\theta_1\theta_2,\ \cdots,\ \theta_{n-1}(1-\alpha_{n,k})]\} \qquad \sum_{k=0}^{\infty} t_k = \infty$$

$$b_n = \alpha_{1,k}M[r_{1,k}/\alpha_{1,k}+\theta_1(1-\alpha_{2,k}+r_{2,k}+\cdots+\alpha_{2,k},\ \cdots,\ \alpha_{n-1,k}\theta_1\theta_2,\ \cdots,\ \theta_{n-1}(1-\alpha_{n,k}+r_{n,k})]=o(t_k)$$

Lemma 2 in [2].

Then we have: $\| x_{1,k+1}-x_1^* \| \to 0$, that is $x_{1,k} \to x_1^*$ $(k\to\infty)$. By letting $k\to\infty$ in (10), then $x_{n,k} \to x_n^*$, $k\to\infty$ similarly, we have that

$$x_{i,k} \to x_i^* \qquad \forall i = 1,2,\cdots,n$$

Above all, we have $x_{i,k} \to x_i^*$, $\forall i=1,2,\cdots,n$. Now, we base on the algorithm (2), we may get the following results similarly.

**Theorem 2** Let $H$ be a real Hilbert space and $K_i$, $i=1,2,\cdots,n$, be nonempty closed convex subsets of $H$. Let $T_i: H \longrightarrow H$ be $\gamma_i$ -strongly monotonic and $\mu_i$ -Lipschitz continuous mappings, for $i=1,2,\cdots,n$. Suppose that $x_i^* \in K_i$, $i=1,2,\cdots,n$ form a solution to the NNVI problems, the sequences $\{x_{1,k+1}\}$, $\{x_{2,k}\}$, $\cdots$, $\{x_{n,k}\}$ are generated by Algorithm 2 and $\{\alpha_{i,k}\}$, $\{r_{i,k}\} \subset [0,1]$, $i=1,2,\cdots,n$, and if the following holds true

(i) $\alpha_{i,k}\to 1$, $r_{i,k}\to 0$, $\sum_{k=0}^{\infty} r_{i,k} < \infty$, $\sum_{k=0}^{\infty}\alpha_{i,k}=\infty$ $(k\to\infty)$.

(ii) $0<\rho_i<2\gamma_i/\mu_i^2$, $i=1,2,\cdots,n$.

(iii) $\mu_{i,k}$ is bounded sequences in $K_i$, $i=1,2,\cdots,n$.

Then $x_{1,k+1}\to x_1^*$, $x_{2,k}\to x_2^*$, $\cdots$, $x_{n,k}\to x_n^*$ $(k\to\infty)$.

**Proof** The way of the proof of this theorem is absolutely the same that in theorem (1).

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# 一种 *n* 步迭代算法的收敛性分析及其应用


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**摘要:** 引入并研究了一类新的非线性变分不等式问题, 给出了一种新的 $n$ 步迭代算法, 并证明了运用此种算法来求解此类变分不等式问题的收敛性.



**关键词:** 广义 $n$ 步迭代算法; 非线性变分不等式; 收敛性

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