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Periodic Solution for a Class of Subquadratic Second-Order Hamiltonian Systems①

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Abstract: Some solvability conditions of periodic solution are obtained for a class of subquadratic Hamiltonian systems by the minimax methods in critical point theory.

Key words: subquadratic Hamiltonian systems; solvability conditions; periodic solution

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Consider the second-order Hamiltonian systems

$$\begin{cases} -\ddot{u}(t) + A(t)u = \nabla F(t, u(t)) & \text{a. e. } t \in [0, T] \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{cases} \tag{1}$$

where $A(t)$ is a continuous symmetric matrix of order N and F: $R \times R^N \longrightarrow R$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for all $x \in R^N$, continuously differentiable in x for a. e. $t \in [0, T]$, and there exist $a \in C(R^+, R^+)$ and $b \in L^1(0, T; R^+)$ such that

$$|F(t, x)| \leqslant a(|x|)b(t) \qquad |\nabla F(t, x)| \leqslant a(|x|)b(t)$$

for all $x \in R^N$ and a. e. $t \in [0, T]$.

With the aids of the variational methods, the existence of periodic solutions for systems (1) is obtained in [1-9]. In the case $A(t) = 0$, there are many solvability conditions for systems (1), such as the coercive type potential condition (see [1]), the convex and subconvex type potential condition (see [2-4]), the even type potential condition (see [5]), the subquadratic potential condition (see [6-8]). Rescently, [7] has obtained the existence of periodic solution for systems (1) under the subquadratic condition where $A(t) = 0$. In this paper, we will obtain the existence of periodic solution for systems (1) under the subquadratic condition. Obviously, Theorem 1 in the paper greatly extends Theorem 2 in [7] which is a special case of our Theorem 1, and Theorem 2 extends Theorem 3 in [8] because the subquadratic potential condition proposed in the following in the paper is more general than that one in [8]. Our main results are the following theorems:

Theorem 1　Suppose that F satisfies assumption (A). If the linear second order systems

$$\begin{cases} -\ddot{u}(t) + A(t)u = \lambda_{k-1} u(t) & \text{a. e. } t \in [0, T] \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{cases} \tag{2}$$

has a nonzero solution assume that

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$$\lim_{|x|\to\infty}[x\cdot\nabla F(t,x)-2F(t,x)]=-\infty \qquad \text{a. e. } t\in[0,T] \tag{3}$$

Assume that there exist consecutive eigenvalues λ_{k-1}, λ_k of the operator $-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+A(t)$ such that

$$\lambda_{k-1}\leqslant\liminf_{|x|\to\infty}\frac{2F(t,x)}{|x|^2}\leqslant\limsup_{|x|\to\infty}\frac{2F(t,x)}{|x|^2}<\lambda_k \qquad \text{a. e. } t\in[0,T] \tag{4}$$

Then there exists a T- periodic solution of systems (1).

Theorem 2 Suppose that F satisfies assumption (A) and the following condition:

$$\lim_{|x|\to\infty}[x\cdot\nabla F(t,x)-2F(t,x)]=+\infty \qquad \text{a. e. } t\in[0,T] \tag{5}$$

Assume that there exist consecutive eigenvalues λ_{k-1}, λ_k of the operator $-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+A(t)$ such that

$$\lambda_{k-1}<\liminf_{|x|\to\infty}\frac{2F(t,x)}{|x|^2}\leqslant\limsup_{|x|\to\infty}\frac{2F(t,x)}{|x|^2}\leqslant\lambda_k \qquad \text{a. e. } t\in[0,T] \tag{6}$$

Then there exists a T- periodic solution of systems (1).

Now, we give some preliminaries.

It is well known that u is a T- periodic solution of system (1) if and only if u is a critical point of the functional φ in H_T^1, where

$$H_T^1=\left\{u:[0,T]\longmapsto R^N \,\middle|\, \begin{array}{l} u \text{ is absolutely continuous,} \\ u(0)=u(T) \text{ and } \dot{u}\in L^2(0,T;R^N)\end{array}\right\}$$

is a Hillbert space with the norm $\|u\|=\left(\int_0^T|u|^2\mathrm{d}t+\int_0^T|\dot{u}|^2\mathrm{d}t\right)^{\frac{1}{2}}$, for $u\in H_T^1$, and

$$\varphi(u)=\frac{1}{2}\int_0^T[|\dot{u}|^2+(A(t)u,u)]\mathrm{d}t-\int_0^T F(t,u)\mathrm{d}t \qquad u\in H_T^1$$

It follows from Theorem 1.4 in [9] that $\varphi\in C^1(H_T^1,R)$ and

$$\langle\varphi'(u),v\rangle=\int_0^T[(\dot{u},\dot{v})+(A(t)u,v)]\mathrm{d}t-\int_0^T(\nabla F(t,u),v)\mathrm{d}t \qquad u,v\in H_T^1$$

where $(\cdot,\cdot)$ and $|\cdot|$ are the usual inner product and norm of R^N.

Lemma 1 Suppose (A), (2) and (3) hold. Then function φ satisfies condition (C).

Proof By contradiction, let $c\in R$ and suppose $\{u_n\}\subset H_T^1$ such that $\|u_n\|\to\infty$, $\varphi(u_n)\to c$ and $(1+\|u_n\|)\|\varphi'(u_n)\|\to 0$ as $n\to\infty$. It follows that

$$\lim_{n\to\infty}[\langle\varphi'(u_n),u_n\rangle-2\varphi(u_n)]=2c$$

More precisely, we have

$$\lim_{n\to\infty}\int_0^T[(\nabla F(t,u_n),u_n)-2F(t,u_n)]\mathrm{d}t=2c \tag{7}$$

Put $z_n=\frac{u_n}{\|u_n\|}$, we have $\|z_n\|=1$, without loss of generality we may assume that: $z_n\rightharpoonup z$ weakly in H_T^1, $z_n\to z$ strongly in $L^2(0,T)$ and $z_n(t)\to z(t)$ for a. e. $t\in[0,T]$.

Let $\varepsilon_1=\left(\frac{1}{2}\lambda_k-\limsup_{|x|\to\infty}\frac{F(t,x)}{|x|^2}\right)/2>0$, by (3), there exists $\delta>0$, such that

$$F(t,x)\leqslant\frac{1}{2}(\lambda_k-\varepsilon_1)|x|^2 \qquad |x|\geqslant\delta, \text{ a. e. } t\in[0,T]$$

By assumption (A), it's easy to know that $|F(t,x)|\leqslant\max_{s\in[0,\delta]}a(s)b(t)$ for all $|x|\leqslant\delta$ and a. e. $t\in[0,T]$. Hence, we obtain

$$F(t,x)\leqslant\frac{1}{2}(\lambda_k-\varepsilon_1)|x|^2+g(t) \tag{8}$$

for all $x\in R^N$ and a. e. $t\in[0,T]$ where $g(t)=\max_{s\in[0,\delta]}a(s)b(t)+\frac{1}{2}|\lambda_k|\delta^2$. By (8), we have

$$\begin{aligned}\varphi(u_n) &= \frac{1}{2}\int_0^T[|\dot{u}_n|^2+(A(t)u_n,u_n)]\mathrm{d}t-\int_0^T F(t,u_n)\mathrm{d}t\\ &=\frac{1}{2}\|u_n\|^2+\frac{1}{2}\int_0^T[(A(t)u_n,u_n)-|u_n|^2]\mathrm{d}t-\int_0^T F(t,u_n)\mathrm{d}t\\ &\geqslant\frac{1}{2}\|u_n\|^2-\frac{1}{2}(c_1+1)\int_0^T|u_n|^2\mathrm{d}t-\frac{1}{2}(\lambda_k-\varepsilon_1)\int_0^T|u_n|^2\mathrm{d}t-\|g\|_{L^1}\\ &\geqslant\frac{1}{2}\|u_n\|^2-\frac{1}{2}(c_1+1)\int_0^T|u_n|^2\mathrm{d}t-\frac{1}{2}\lambda_k\int_0^T|u_n|^2\mathrm{d}t-\|g\|_{L^1}\end{aligned}$$

where $c_1=\max\limits_{t\in[0,T]}|A(t)|$. Therefore, one obtains

$$\frac{\varphi(u_n)}{\|u_n\|^2}\geqslant\frac{1}{2}(1-(c_1+1+\lambda_k)\|z_n\|_{L^2}^2)-\frac{\|g\|_{L^1}}{\|u_n\|^2}$$

Passing to the limit in the inequality, by using $\varphi(u_n)\to c$ as $n\to\infty$, we obtain

$$\frac{1}{2}[1-(c_2+1+\lambda_k)\|z\|_{L^2}^2]\leqslant 0$$

which implies that $z\neq 0$.

Now by (2), there exists $\delta_0>0$ such that $(\nabla F(t,x),x)-2F(t,x)\leqslant 0$ for all $|x|\geqslant\delta_0$ and a. e. $t\in[0,T]$ and by assumption (A) we have $(\nabla F(t,x),x)-2F(t,x)\leqslant c_2 b(t)$ for all $|x|\leqslant\delta_0$ and a. e. $t\in[0,T]$ where $c_2=(2+\delta_0)\max\limits_{s\in[0,\delta_0]}a(s)b(t)$. So we get

$$(\nabla F(t,x),x)-2F(t,x)\leqslant c_2 b(t)$$

for all $x\in R^N$ and a. e. $t\in[0,T]$. Hence we get

$$\begin{aligned}&\int_0^T[(\nabla F(t,u_n),u_n)-2F(t,u_n)]\mathrm{d}t\\ =&\int_{\{t:\,z(t)\neq 0\}}[(\nabla F(t,u_n),u_n)-2F(t,u_n)]\mathrm{d}t+\int_{[0,T]\setminus\{t:\,z(t)\neq 0\}}[(\nabla F(t,u_n),u_n)-2F(t,u_n)]\mathrm{d}t\\ \leqslant&\int_{\{t:\,z(t)\neq 0\}}[(\nabla F(t,u_n),u_n)-2F(t,u_n)]\mathrm{d}t+\int_{[0,T]\setminus\{t:\,z(t)\neq 0\}}c_2\,b(t)\mathrm{d}t\end{aligned}$$

An application of Fatou's lemma yields

$$\int_0^T[(\nabla F(t,u_n),u_n)-2F(t,u_n)]\mathrm{d}t\to-\infty\qquad n\to\infty$$

which is a contradiction to (7). In a similar way to Proposition 4.3 in [9], we can prove that $\{u_n\}$ has a convergent subsequence. Thus, the proof of Lemma 1 is complete.

Now we give the proofs of the main results.

Proof of Theorem 1 Let $H_T^1=H^0\oplus H^+\oplus H^-$. where $H^+\triangleq_{j>k-1}\mathrm{Ker}\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+A(t)-\lambda_j I\right)$, $H^-\triangleq_{j<k-1}\mathrm{Ker}\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+A(t)-\lambda_j I\right)$, $H^0=\mathrm{Ker}\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+A(t)-\lambda_{k-1}I\right)$. By the Saddle Point Theorem (see Theorem 2.13 in [10]), we only need to prove

(l_1) $\inf\limits_{H^+}\varphi(u)\geqslant d$, where d is a constant.

(l_2) $\sup\limits_{H^-\oplus H^0}\varphi(u)<+\infty$.

It follows from (8) that

$$\begin{aligned}\varphi(u)&=\frac{1}{2}\int_0^T[|\dot{u}|^2+(A(t)u,u)]\mathrm{d}t-\int_0^T F(t,u)\mathrm{d}t\\ &\geqslant\frac{1}{2}\int_0^T[|\dot{u}|^2+(A(t)u,u)]\mathrm{d}t-\frac{1}{2}(\lambda_k-\varepsilon_1)\|u\|_{L^2}^2-\|g\|_{L^1}\\ &\geqslant\frac{1}{2}\lambda_k\|u\|_{L^2}^2-\frac{1}{2}(\lambda_k-\varepsilon_1)\|u\|_{L^2}^2-\|g\|_{L^1}\geqslant\frac{1}{2}\varepsilon_1\|u\|_{L^2}^2-\|g\|_{L^1}\geqslant-\|g\|_{L^1}\end{aligned}$$

for all $u\in H^+$, which implies (l_1).

Now, we prove (l_2). Let's suppose that $G(t, x) = F(t, x) - \frac{1}{2}\lambda_{k-1} |x|^2$. By (2), we obtain, for every $\beta > 0$, there exists $\delta_1 > 0$ such that

$$x \cdot \nabla G(t, x) - 2G(t, x) \leqslant -2\beta \qquad |x| \geqslant \delta_1, \text{ a. e. } t \in [0, T] \tag{9}$$

Let $s \geqslant 1$, using (9) and integrating the relation

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{G(t, sx)}{s^2}\right] = \frac{sx \cdot \nabla G(t, sx) - 2G(t, sx)}{s^3} \leqslant \frac{-\beta}{s^3}$$

over an interval $[1, S] \subset [1, \infty)$, we get

$$\frac{G(t, Sx)}{S^2} - G(t, x) \leqslant \beta\left[\frac{1}{S^2} - 1\right]$$

Since $\liminf_{S\to\infty} \frac{G(t, Sx)}{S^2} \geqslant 0$ by (3), we obtain $G(t, x) \geqslant \beta$ for all $|x| \geqslant \delta_1$ and a. e. $t \in [0, T]$. By assumption (A), we can get

$$|G(t, x)| = \left|F(t, x) - \frac{1}{2}\lambda_{k-1} |x|^2\right| \leqslant \max_{s\in[0, \delta_1]} a(s)\, b(t) + \frac{1}{2} |\lambda_{k-1}| \delta_1^2$$

for all $|x| \leqslant \delta_1$ and a. e. $t \in [0, T]$. That's

$$G(t, x) = F(t, x) - \frac{1}{2}\lambda_{k-1} |x|^2 \geqslant \beta - \max_{s\in[0, \delta_1]} a(s)b(t) + \frac{1}{2} |\lambda_{k-1}| \delta_1^2 \tag{10}$$

for all $x \in R^N$ and a. e. $t \in [0, T]$.

For all $u = u^- + u^0 \in H^- \oplus H^0$, we have

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\int_0^T [|\dot{u}|^2 + (A(t)u, u)]\mathrm{d}t - \frac{1}{2}\lambda_{k-1}\int_0^T |u|^2 \mathrm{d}t - \int_0^T \left[F(t, u) - \frac{1}{2}\lambda_{k-1} |u|^2\right]\mathrm{d}t \\ &\leqslant -\int_0^T \left[F(t, u) - \frac{1}{2}\lambda_{k-1} |u|^2\right]\mathrm{d}t \leqslant \max_{s\in[0, \delta_1]} a(s) \|b\|_{L^1} + \frac{1}{2} |\lambda_{k-1}| \delta_1^2\end{aligned}$$

Hence (l_2) holds. Now, the proof of Theorem 1 is over.

Proof of Theorem 2 According to theory of spectrum, we get $H_T^1 = H^0 \oplus H^+ \oplus H^-$, where

$$H^+ \triangleq_{j>k} \mathrm{Ker}\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + A(t) - \lambda_j I\right)$$

$$H^- \triangleq_{j<k} \mathrm{Ker}\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + A(t) - \lambda_j I\right)$$

$$H^0 = \mathrm{Ker}\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + A(t) - \lambda_k I\right)$$

Let $\varepsilon_2 = \left(\inf_{[0, T]} \liminf_{|x|\to\infty} \frac{F(t, x)}{|x|^2} - \frac{1}{2}\lambda_{k-1}\right)/2 > 0$. In a way similar to the proof of (8), we can prove that

$$\frac{1}{2}(\lambda_{k-1} + \varepsilon_2) |x|^2 - g'(t) \leqslant F(t, x) \leqslant \frac{1}{2}\lambda_k |x|^2 + g(t) \tag{11}$$

for all $x \in R^N$ and a. e. $t \in [0, T]$ by (6), where $g'(t) \in L^1(0, T)$. In a way similar to the proof of Lemma 1, we can prove that φ satisfies condition (C) by (11). As the same as in the proof of Theorem 1, we only need to prove

(l_3) $\inf_{H^+\oplus H^0} \varphi(u) \geqslant d$. where d is a constant.

(l_4) $\varphi(u) \to -\infty$, as $\|u\| \to \infty$ in H^-.

It follows from (11) that

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\int_0^T [|\dot{u}|^2 + (A(t)u, u)]\mathrm{d}t - \int_0^T F(t, u)\mathrm{d}t \\ &\geqslant \frac{1}{2}\int_0^T [|\dot{u}|^2 + (A(t)u, u)]\mathrm{d}t - \frac{1}{2}\lambda_k \|u\|_{L^2}^2 - \|g\|_{L^1} \geqslant -\|g\|_{L^1}\end{aligned}$$

for all $u \in H^+ \oplus H^0$, which implies (l_3). It follows from (11) that

$$\varphi(u)=\frac{1}{2}\int_0^T[|\dot{u}|^2+(A(t)u,u)]\mathrm{d}t-\int_0^T F(t,u)\mathrm{d}t$$

$$\leqslant\frac{1}{2}\lambda_{k-1}\int_0^T|u|^2\mathrm{d}t-\frac{1}{2}(\lambda_{k-1}+\varepsilon_2)\int_0^T|u|^2\mathrm{d}t+\|g'\|_{L^1}\leqslant-\frac{\varepsilon_2}{2}\int_0^T|u|^2\mathrm{d}t+\|g'\|_{L^1}$$

for all $u\in H^-$. Since $\dim(H^-)<\infty$, there exists $C>0$ such that $\|u\|\leqslant C\|u\|_{L^2}$ for all $u\in H^-$. Hence we get that $\varphi(u)\to-\infty$ as $\|u\|\to\infty$. We complete the proof of Theorem 2.

Now, we give the examples to show there are functions satisfying our Theorems and not satisfying the corresponding results in [1-8].

Let $A(t)$ be a continuous symmetric matrix of order N and

$$F(t,x)=(1+|x|^2)^{\frac{1}{2}}\ln(1+|x|^2)-\frac{1}{2}\lambda|x|^2+(x,e(t))$$

for all $x\in R^N$ and $t\in[0,T]$, where $\lambda_{k-1}\leqslant\lambda<\lambda_k$ and $e(t)\in L^\infty(0,T;R^N)$. We know the functions A and F satisfy our Theorem 1 and not satisfying the corresponding results in [1-8]. Let

$$F(t,x)=\frac{1}{2}\lambda|x|^2-(1+|x|^2)^{\frac{1}{2}}\ln(1+|x|^2)-(x,e(t))$$

for all $x\in R^N$ and $t\in[0,T]$, where $\lambda_{k-1}<\lambda\leqslant\lambda_k$ and $e(t)\in L^\infty(0,T;R^N)$. The functions F satisfy our Theorem 2 and not satisfy the corresponding results in [1-8].

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一类次二次 Hamilton 系统的周期解

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摘要： 运用临界点理论的极小极大方法得到一类次二次 Hamilton 系统的周期解的可解性条件.

关键词： 次二次哈密尔顿系统；可解性条件；周期解

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