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# Existence of Solutions for a Class of Semilinear Elliptic Equations①

OU Zeng-qi,　TANG Chun-lei

*School of Mathematics and Statistics, Southwest University, Chongqing 400715, China*

**Abstract**: Existence of solutions for a class of semilinear elliptic equations is obtained by the minimax methods in critical point theory.

**Key words**: semilinear; elliptic equation; critical point theory

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We consider the elliptic boundary value problem

$$-\triangle u + b(x)u = f(x, u) \qquad x \in \Omega,\ u = 0 \text{ on } \partial\Omega \tag{1}$$

where $\Omega \subset R^N (N \geqslant 1)$ is a bounded domain whose boundary is a smooth manifold, $b \in L^\infty(\Omega)$ and $f \in C(\bar{\Omega} \times R, R)$. Setting $F(x, t) = \int_0^t f(x, s)\mathrm{d}s$ for every $t \in R$ and a. e. $x \in \Omega$, we suppose that function $F$ satisfies the following superquadratic condition:

$$\frac{F(x, t)}{t^2} \to +\infty \qquad |t| \to \infty,\ x \in \Omega \tag{2}$$

[1] applied the local linking to problem (1) under the following superquadratic condition(the Ambrosetti-Rabinowitz-type condition): there are $\mu > 2$ and $L > 0$ such that

$$0 < \mu F(x, t) \leqslant f(x, t)t \tag{3}$$

for $|t| \geqslant L$ and $x \in \Omega$.

It is easy to see that the function

$$F(x, t) = t^2(\ln(t^4 + 1))^3 \qquad \forall (x, t) \in \Omega \times R$$

does satisfy condition (2) but not satisfy (3). In this paper, we shall study the existence of solution for problem (1) under the condition (2).

Let $\lambda_k$ $(k = 1, 2, \cdots)$ be the kth distinct eigenvalue of the eigenvalue problem

$$-\triangle u + b(x)u = \lambda u \qquad x \in \Omega,\ u = 0 \text{ on } \partial\Omega$$

**Theorem 1**　Suppose that f satisfies (2) and the following conditions:

(f1) There exist constants $a_1 > 0$, $a_2 \geqslant 0$ such that

$$|f(x, t)| \leqslant a_1 |t|^{p-1} + a_2$$

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作者简介：欧增奇(1975－)，男，重庆铜梁人，讲师，主要从事泛函分析方面的研究.

for every $t \in R$ and a. e. $x \in \Omega$, where $2 < p < \frac{2N}{N-2}$ if $N \geqslant 3$ and $p$ may be replaced by any number in $(2, +\infty)$ if $N = 1$ or $N = 2$;

(f2) There exist constants $\beta > \frac{2N(p-1)}{N+2}$ (if $N \geqslant 3$), $\beta > p-1$ (if $N = 1, 2$), $c_1 > 0$ and $r > 0$ such that

$$tf(x, t) - 2F(x, t) \geqslant c_1 |t|^{\beta} \qquad \forall |t| \geqslant r, \ \forall x \in \Omega$$

(f3) There exist constants $m \geqslant 1$, $b > 0$ and $\delta > 0$ such that

$$\lambda_m \leqslant \frac{f(x, t)}{t} \leqslant \lambda_{m+1} - b$$

for every $0 < |t| \leqslant \delta$ and a. e. $x \in \Omega$. Then problem (1) has at least one nonzero solution.

**Remark** There are functions $f$ satisfying Theorem 1 and not satisfying Theorem 4 in [1]. For example, let

$$f(x, t) = \begin{cases} \lambda_1 t & |t| \leqslant 1 \\ 2c\left(t\ln(1+t^2) + \dfrac{t^3}{1+t^2}\right) & |t| \geqslant 1 \end{cases}$$

where $c = \frac{\lambda_1}{2\ln 2 + 1}$ is a constant such that $f(x, t)$ is continuous in $|t| = 1$. Then $f(x, t)$ satisfies the conditions of Theorem 1, but $f(x, t)$ does not satisfy the conditions of Theorem 4 in [1].

In this paper, we shall use Theorem 2 in [1] to prove our result. Let $X$ be a real Banach space with $X = X^1 \oplus X^2$ and $X_0^j \subset X_1^j \subset \cdots \subset X^j$ such that $X^j = \overline{\bigcup_{n \in N} X_n^j}$, $j = 1, 2$. For every multi-index $\alpha = (\alpha_1, \alpha_2) \in N^2$, let $X_\alpha = X_{\alpha_1}^1 \oplus X_{\alpha_2}^2$. We know that $\alpha \leqslant \beta$ is equivalent to $\alpha_1 \leqslant \beta_1$, $\alpha_2 \leqslant \beta_2$. A sequence $(\alpha_n) \subset N^2$ is admissible, if for every $\alpha \in N^2$ there is $m \in N$ such that $n \geqslant m$, we obtain that $\alpha_n \geqslant \alpha$. We say that $\varphi \in C^1(X, R)$ satisfies the $(\mathrm{PS})^*$ condition if every sequence $(\alpha_n)$ is admissible and satisfies

$$u_{\alpha_n} \in X_{\alpha_n} \qquad c = \sup \varphi(u_{\alpha_n}) < \infty \qquad \varphi'_{\alpha_n}(u_{\alpha_n}) \to 0$$

then $(u_{\alpha_n})$ contains a convergent subsequence, where $\varphi_\alpha = \varphi_{|X_\alpha}$.

Let

$$\varphi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x + \frac{1}{2}\int_\Omega b(x) u^2 \mathrm{d}x - \int_\Omega F(x, u) \mathrm{d}x \tag{4}$$

for $u \in H_0^1(\Omega)$, where $H_0^1(\Omega)$ is a Hilbert space with the norm given by

$$\| u \| = \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}}$$

Under the conditions of Theorem 1, $\varphi \in C^1(H_0^1(\Omega), R)$ and it is well known that $u \in H_0^1(\Omega)$ is a solution of problem (1) if and only if $u$ is a critical point of $\varphi$.

**Proof of Theorem 1** Let $X = H_0^1(\Omega)$, $X^2 = E(\lambda_1) \oplus \cdots \oplus E(\lambda_m)$, where $E(\lambda_k)(k = 1, 2, \cdots)$ is the eigenspace corresponding to $\lambda_k$, we obtain that $X^2$ is a finite dimensional subspace of $X$, and $X^1 = (X^2)^\perp$. We choose $(e_n)_{n \geqslant 1}$ to be an orthonormal basis of $X^1$, defining

$$X_n^1 = \mathrm{span}\{e_1, e_2, \cdots, e_n\} \qquad n \in N$$
$$X_n^2 = X^2 \qquad n \in N$$
$$X^j = \overline{\bigcup_{n \in N} X_n^j} \qquad j = 1, 2$$

(a) We claim that $\varphi$ satisfies the local linking at zero with respect to $(X^1, X^2)$, i. e. there is $\delta_0 > 0$ such that

$$\varphi(u) \leqslant 0 \qquad \forall u \in X^2, \ \| u \| \leqslant \delta_0$$

$$\varphi(u) \geqslant 0 \qquad \forall u \in X^1, \ \| u \| \leqslant \delta_0$$

Indeed from (f3) we get

$$\lambda_m t^2 \leqslant t f(x, t) \leqslant (\lambda_{m+1} - b) t^2$$

for every $0 < | t | \leqslant \delta$ and a. e. $x \in \Omega$, moreover ones see that

$$\lambda_m s t^2 \leqslant t f(x, st) \leqslant (\lambda_{m+1} - b) s t^2$$

for every $0 < s \leqslant 1$, $0 < | t | \leqslant \delta$ and a. e. $x \in \Omega$. By the definition of $F(x, t)$, we have

$$\frac{1}{2}\lambda_m t^2 \leqslant F(x, t) \leqslant \frac{1}{2}(\lambda_{m+1} - b) t^2$$

for every $0 < | t | \leqslant \delta$ and a. e. $x \in \Omega$. By the equivalence of the norms on the finite dimensional space $X^2$, there exists $C_1 > 0$ such that

$$\| u \|_\infty \leqslant C_1 \| u \|$$

for all $u \in X^2$. So we obtain

$$\varphi(u) \leqslant \frac{1}{2}\int_\Omega | \nabla u |^2 \mathrm{d}x + \frac{1}{2}\int_\Omega b(x) u^2 \mathrm{d}x - \frac{1}{2}\lambda_m \int_\Omega | u |^2 \mathrm{d}x \leqslant 0$$

for all $u \in X^2$ with $\| u \| \leqslant \frac{\delta_0}{C_1}$.

By (f1) we get

$$| F(x, t) | \leqslant C_2 (p^{-1} | t |^p + | t |)$$

for every $t \in R$ and a. e. $x \in \Omega$. Hence we obtain

$$| F(x, t) | \leqslant d_1 (p^{-1} + \delta^{1-p}) | t |^p = d_2 | t |^p$$

for every $| t | \geqslant \delta$ and a. e. $x \in \Omega$. So we get

$$F(x, t) \leqslant \frac{1}{2}(\lambda_{m+1} - b) t^2 + d_2 | t |^p$$

for every $t \in R$ and a. e. $x \in \Omega$. Now for all $u \in X_1$, it follows that

$$\begin{aligned} \varphi(u) &\geqslant \frac{1}{2}\int_\Omega | \nabla u |^2 \mathrm{d}x + \frac{1}{2}\int_\Omega b(x) u^2 \mathrm{d}x - \frac{1}{2}(\lambda_{m+1} - b)\int_\Omega u^2 \mathrm{d}x - d_2 \| u \|_{L^p}^p \\ &\geqslant \frac{b}{2\lambda_{m+1}} \| u \|^2 - d_2 C^p \| u \|^p \end{aligned}$$

where $C$ is a positive constant such that $\| u \|_{L^p} \leqslant C \| u \|$ for all $u \in H_0^1(\Omega)$. Let

$$\| u \| \leqslant \left[ \frac{b}{2\lambda_{m+1} d_2 C^p} \right]^{\frac{1}{p-2}}$$

for all $u \in X^1$, we get

$$\varphi(u) \geqslant 0$$

(b) We claim that $\varphi$ satisfies the $(\mathrm{PS})^*$ condition. Let $(u_{\alpha_n})$ be a sequence such that $(\alpha_n)$ is admissible and satisfies

$$u_{\alpha_n} \in X_{\alpha_n} \qquad c = \sup \varphi(u_{\alpha_n}) < \infty \qquad \varphi'_{\alpha_n}(u_{\alpha_n}) \to 0$$

We prove that $(u_{\alpha_n})$ is bounded sequence in $X$, then by a standard argument, $(u_{\alpha_n})$ has a convergent subsequence. For otherwise, going if necessary to a subsequence, we can assume that $\| u_{\alpha_n} \| \to \infty$ as $n \to \infty$. By (f2), there exist $c_2 > 0$, $c_3 > 0$ such that

$$t \cdot f(x, t) - 2F(x, t) \geqslant c_2 | t |^\beta - c_3 \qquad \forall (x, t) \in \Omega \times R$$

So we obtain

$$\begin{aligned} 2\varphi(u_{\alpha_n}) - \langle \varphi'_{\alpha_n}(u_{\alpha_n}), u_{\alpha_n} \rangle &= \int_\Omega [u_{\alpha_n} \cdot f(x, u_{\alpha_n}) - 2F(x, u_{\alpha_n})] \mathrm{d}x \\ &\geqslant c_2 \int_\Omega | u_{\alpha_n} |^\beta \mathrm{d}x - c_3 | \Omega | \end{aligned}$$

hence we have

$$\frac{\int_\Omega |u_{\alpha_n}|^\beta \mathrm{d}x}{\|u_{\alpha_n}\|} \to 0 \qquad n\to\infty \tag{5}$$

Let $u_{\alpha_n} = u_{\alpha_n}^- + u_{\alpha_n}^0 + u_{\alpha_n}^+ \in E^- \oplus E^0 \oplus E^+$, by (f1), $\frac{\beta}{\beta-p+1} < \frac{2N}{N-2}$ $(N\geqslant 3)$ and the Hölder's inequality, one has

$$\begin{aligned}
\langle \varphi'_{\alpha_n}(u_{\alpha_n}), u_{\alpha_n}^+\rangle &= \|u_{\alpha_n}^+\|^2 + \int_\Omega b(x) u_{\alpha_n} u_{\alpha_n}^+ \mathrm{d}x - \int_\Omega u_{\alpha_n}^+ \cdot f(x, u_{\alpha_n})\mathrm{d}x \\
&\geqslant \|u_{\alpha_n}^+\|^2 - \|b\|_{L^\infty(\Omega)} \|u_{\alpha_n}\|_{L^2} \|u_{\alpha_n}^+\|_{L^2} - \int_\Omega |f(x, u_{\alpha_n})| \cdot |u_{\alpha_n}^+| \mathrm{d}x \\
&\geqslant \|u_{\alpha_n}^+\|^2 - C\|u_{\alpha_n}\| \|u_{\alpha_n}^+\| - \int_\Omega [a_1 |u_{\alpha_n}|^{p-1} |u_{\alpha_n}^+| + a_2 |u_{\alpha_n}^+|]\mathrm{d}x \\
&\geqslant \|u_{\alpha_n}^+\|^2 - c_4 \left(\int_\Omega (|u_{\alpha_n}|^{p-1})^{\frac{\beta}{p-1}}\mathrm{d}x\right)^{\frac{p-1}{\beta}} \left(\int_\Omega |u_{\alpha_n}^+|^{\frac{\beta}{\beta+1-p}}\mathrm{d}x\right)^{\frac{\beta-p+1}{\beta}} - \\
&\quad c_5\|u_{\alpha_n}^+\| - C\|u_{\alpha_n}\| \|u_{\alpha_n}^+\| \\
&\geqslant \|u_{\alpha_n}^+\|^2 - c_5\|u_{\alpha_n}^+\| - c_6 \|u_{\alpha_n}\|_{L^\beta}^{p-1} \cdot \|u_{\alpha_n}^+\| - C\|u_{\alpha_n}\| \|u_{\alpha_n}^+\|
\end{aligned}$$

where $c_4, c_5, c_6 > 0$. Noting that $1 < p-1 < \beta$, by (5), we have

$$\frac{\|u_{\alpha_n}^+\|}{\|u_{\alpha_n}\|} \to 0 \qquad n\to\infty \tag{6}$$

Similarly for $u_{\alpha_n}^-$, we also get

$$\frac{\|u_{\alpha_n}^-\|}{\|u_{\alpha_n}\|} \to 0 \qquad n\to\infty \tag{7}$$

By (f2), there exist $c_7 > 0$ and $c_8 > 0$ such that

$$t \cdot f(x, t) - 2F(x, t) \geqslant c_7 |t| - c_8 \qquad \forall (x, t) \in \Omega \times R$$

which implies that

$$\begin{aligned}
2\varphi(u_{\alpha_n}) - \langle \varphi'_{\alpha_n}(u_{\alpha_n}), u_{\alpha_n}\rangle &= \int_\Omega [u_{\alpha_n} \cdot f(x, u_{\alpha_n}) - 2F(x, u_{\alpha_n})]\mathrm{d}x \\
&\geqslant \int_\Omega [c_7 |u_{\alpha_n}| - c_8]\mathrm{d}x \\
&\geqslant \int_\Omega [c_7 |u_{\alpha_n}^0| - c_7 |u_{\alpha_n}^+| - c_7 |u_{\alpha_n}^-| - c_8]\mathrm{d}x \\
&\geqslant c_9 \|u_{\alpha_n}^0\| - c_{10}(\|u_{\alpha_n}^+\| + \|u_{\alpha_n}^-\| + 1)
\end{aligned}$$

By (6) and (7), we have

$$\frac{\|u_{\alpha_n}^0\|}{\|u_{\alpha_n}\|} \to 0 \qquad n\to\infty$$

Hence combined this inequality with (6) and (7) yields to

$$1 = \frac{\|u_{\alpha_n}\|}{\|u_{\alpha_n}\|} \leqslant \frac{\|u_{\alpha_n}^-\| + \|u_{\alpha_n}^0\| + \|u_{\alpha_n}^+\|}{\|u_{\alpha_n}\|} \to 0$$

as $n\to\infty$, a contradiction. So $(u_{\alpha_n})$ must be bounded. Similar to the proof of Theorem 2.1 in [4], there is a convergent subsequence in $(u_{\alpha_n})$.

(c) We claim that for every $m \in N$,

$$\varphi(u) \to -\infty \qquad \|u\| \to \infty \qquad \forall u \in X_m^1 \oplus X^2$$

Since $X_m^1 \oplus X^2$ is finite dimensional, there exists $C_3 > 0$ such that $\|u\| \leqslant C_3 \|u\|_{L^2}$ for all $u \in X_m^1 \oplus X^2$. By (2), for any $M_1 > 0$, there exists $M_2 > 0$ such that

$$F(x,\ t) \geqslant M_1 t^2 - M_2 \qquad \forall (x,\ t) \in \Omega \times R$$

We have

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x + \frac{1}{2}\int_\Omega b(x)u^2 \mathrm{d}x - \int_\Omega F(x,\ u)\mathrm{d}x \\ &\leqslant \frac{1}{2}\|u\|^2 + \frac{1}{2}\|b\|_{L^\infty}\|u\|_{L^2}^2 - M_1\|u\|_{L^2}^2 + M_2|\Omega| \\ &\leqslant \left(\frac{C_3^2}{2} + \frac{1}{2}\|b\|_{L^\infty} - M_1\right)\|u\|_{L^2}^2 + M_2|\Omega|\end{aligned}$$

Let $M_1$ satisfy $\frac{1}{2}C_3^2 + \frac{1}{2}\|b\|_{L^\infty} - M_1 < 0$, it is easy to conclude that

$$\varphi(u) \to -\infty \qquad \|u\| \to \infty,\ u \in X_m^1 \oplus X^2$$

According to (a),(b),(c), Theorem 1 is proved by Theorem 2 in [1].

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# 一类半线性椭圆方程解的存在性

欧增奇，　唐春雷

西南大学 数学与统计学院，重庆 400715

**摘要：**通过临界点理论中的极小极大方法得到了一个关于一类半线性椭圆方程解存在性的结果.

**关键词：**半线性；椭圆方程；临界点理论

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