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On Locally Dually Flat Matsumoto Metrics with Isotropic S-Curvature[①]

JIANG Jing-nong[1,2]

1. School of Mathematics and Statistics, Chongqing University of Technology, Chongqing 400054, China;
2. Department of Medical Information Engineering, Zunyi Medical College, Zunyi Guizhou 563003, China

Abstract: Matsumoto metrics forms an important class of Finsler metrics in the form of $F=\frac{\alpha^2}{\alpha-\beta}$, where $\alpha=\sqrt{a_{ij}y^iy^j}$ denotes a Riemannian metric and $\beta=b_i(x)y^i$ denotes a 1-form on a manifold. In this paper, the author finds some equations that characterize locally dually flat Matsumoto metrics and classify those with isotropic S-curvature.

Key words: Finsler metric; Matsumoto metric; locally dually flat Finsler metric; S-curvature

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Locally dually flat Finsler metrics arise from information geometry. Such metrics have special geometric properties. In this paper, we characterize locally dually flat Matsumoto metrics, we get the following two Theorems:

Theorem 1 The Matsumoto metric $F=\frac{\alpha^2}{\alpha-\beta}$ on n-dimensional manifold M $(n\geqslant 3)$ is locally dually flat if and only if in an adapted coordinate system β and α satisfy

$$G_\alpha^l=\frac{1}{3}(2\theta-\tau\beta)y^l+\frac{1}{3}(\theta^l+\tau b^l)\alpha^2 \tag{1}$$

$$r_{00}=\frac{2}{3}\theta\beta+\tau\beta^2-\frac{1}{3}(2\theta^l b_l+2\tau b^2+\tau)\alpha^2 \tag{2}$$

$$s_{l0}=\frac{1}{3}\theta_l\beta-\frac{1}{3}\theta b_l+\frac{10}{9}\tau\beta b_l-\frac{10}{9}\tau\frac{\beta^2}{\alpha^2}y_l \tag{3}$$

where $\theta=\theta_k(x)y^k$ is a 1-form and $\tau=\tau(x)$ is a scalar function on M.

If a locally dually flat Matsumoto metric is isotropic S-curvature, then it can be completely determined.

Theorem 2 The Matsumoto metric $F=\frac{\alpha^2}{\alpha-\beta}$ on n-dimensional manifold M $(n\geqslant 3)$ is locally dually flat with isotropic S-curvature if and only if α is flat and β is parallel with respect to α. In this case, F is isometric to a Minkowski metric $\widetilde{F}(y)=\frac{|y|^2}{|y|-b_iy^i}$ with zero S-curvature, where $|\cdot|$ is the Euclidean

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作者简介:蒋经农(1982-),男,重庆忠县人,硕士研究生,主要从事微分几何学的研究.

metric on $\mathbb{R}^n$, and b_i $(i=1,2,\cdots,n)$ is non-zero constant.

Proof of Theorem 1 It is straight forward to verify the sufficient condition. Thus we shall only prove the necessary condition. Plug $F=\dfrac{\alpha^2}{\alpha-\beta}$ into formula (24) in reference [1], we get the following equation:

$$(2y_mG^m_\alpha b_l+3\frac{\partial(y_mG^m_\alpha)}{\partial y^l}\beta-9a_{ml}G^m_\alpha\beta+2b_mG^m_\alpha y_l+r_{00}y_l)\alpha^3+(6a_{ml}G^m_\alpha\beta^2-8\beta y_mG^m_\alpha b_l-8\beta b_mG^m_\alpha y_l-4r_{00}\beta y_l-2\beta^2\frac{\partial(y_mG^m_\alpha)}{\partial y^l})\alpha^2-2\alpha\beta y_mG^m_\alpha y_l+8\beta^2y_mG^m_\alpha y_l=0 \tag{4}$$

equation (4) is equivalent the following two equations:

$$(3s_{l0}-r_{l0}-\frac{\partial(b_mG^m_\alpha)}{\partial y^l})\alpha^4+(2y_mG^m_\alpha b_l+3\frac{\partial(y_mG^m_\alpha)}{\partial y^l}\beta-9a_{ml}G^m_\alpha\beta+2b_mG^m_\alpha y_l+r_{00}y_l)\alpha^2-2\beta y_mG^m_\alpha y_l=0 \tag{5}$$

$$(3a_{ml}G^m_\alpha-\frac{\partial(y_mG^m_\alpha)}{\partial y^l}+\frac{\partial(b_mG^m_\alpha)}{\partial y^l}\beta+3r_{00}b_l-3s_{l0}\beta+r_{l0}\beta+6b_mG^m_\alpha b_l)\alpha^4+8\beta^2y_mG^m_\alpha y_l+(6a_{ml}G^m_\alpha\beta-8y_mG^m_\alpha b_l-8b_mG^m_\alpha y_l-4r_{00}y_l-2\beta\frac{\partial(y_mG^m_\alpha)}{\partial y^l})\beta\alpha^2=0 \tag{6}$$

Contracting equations (5) and (6) with b^l yield

$$(3s_0-r_0-\frac{\partial(b_mG^m_\alpha)}{\partial y^l}b^l)\alpha^4+(2y_mG^m_\alpha b^2+3\frac{\partial(y_mG^m_\alpha)}{\partial y^l}b^l\beta-9b_mG^m_\alpha\beta+2b_mG^m_\alpha\beta+r_{00}\beta)\alpha^2-2\beta y_mG^m_\alpha\beta=0 \tag{7}$$

$$(3b_mG^m_\alpha-\frac{\partial(y_mG^m_\alpha)}{\partial y^l}b^l+\frac{\partial(b_mG^m_\alpha)}{\partial y^l}b^l\beta+3r_{00}b^2-3s_0\beta+r_0\beta+6b_mG^m_\alpha b^2)\alpha^4+8\beta^3y_mG^m_\alpha=0 \tag{8}$$

Multiply formula (7) by β and plus formula (8), we get

$$(3b_mG^m_\alpha-\frac{\partial(y_mG^m_\alpha)}{\partial y^l}b^l)(\alpha^2-\beta^2)\alpha^2=(6\beta y_mG^m_\alpha-3\alpha^2r_{00}-6\alpha^2b_mG^m_\alpha)(b^2\alpha^2-\beta^2) \tag{9}$$

Because $\alpha^2-\beta^2$ and α^2 and $b^2\alpha^2-\beta^2$ are all irreducible polynomials of y^i, Thus there is a function $\tau=\tau(x)$ on M such that

$$6\beta y_mG^m_\alpha-3\alpha^2r_{00}-6\alpha^2b_mG^m_\alpha=\tau(\alpha^2-\beta^2)\alpha^2 \tag{10}$$

$$3b_mG^m_\alpha-\frac{\partial(y_mG^m_\alpha)}{\partial y^l}b^l=\tau(b^2\alpha^2-\beta^2) \tag{11}$$

formula (10) can be reduced into

$$6\beta y_mG^m_\alpha=(\tau\alpha^2+3r_{00}+6b_mG^m_\alpha-\tau\beta^2)\alpha^2 \tag{12}$$

Since α^2 does not contain the factor β, we have

$$y_mG^m_\alpha=\theta\alpha^2 \tag{13}$$

$$b_mG^m_\alpha=\beta\theta-\frac{\tau\alpha^2}{6}+\frac{\tau\beta^2}{6}-\frac{r_{00}}{2} \tag{14}$$

where $\theta=\theta_k(x)y^k$ is a 1-form on M. We obtain

$$\frac{\partial(b_mG^m_\alpha)}{\partial y^l}=\theta_l\beta+\theta b_l-\frac{\tau}{3}y_l+\frac{\tau}{3}\beta b_l-r_{l0} \tag{15}$$

$$\frac{\partial(y_mG^m_\alpha)}{\partial y^l}=\theta_l\alpha^2+2\theta y_l \tag{16}$$

By formulas (13)−(16), formulas (5) and (6) are equivalent to the following two equations:

$$(\tau\beta b_l-9s_{l0}-6\theta_l\beta-3\theta b_l)\alpha^2-18\beta\theta y_l+27a_{ml}G^m_\alpha\beta-\tau\beta^2y_l=0 \tag{17}$$

$$(3\theta_l+3\tau b_l)\alpha^4+(9\beta s_{l0}-9a_{ml}G^m_\alpha+3\beta^2\theta_l+3\beta\theta b_l+6\theta y_l-3\tau\beta y_l-4\tau\beta^2b_l)\alpha^2+12\beta^2\theta y_l+4\tau\beta^3y_l-18\beta^2a_{ml}G^m_\alpha=0 \tag{18}$$

From equations (17) and (18), we obtain formulas (1) and (3). Substituting formula (1) into formula (14), we get formula (2). This completes the proof of Theorem 1.

Proof of Theorem 2 By Theorem 2 in reference [2] and Theorem 1, if F is locally dually flat with isotropic S-curvature, then $\tau=\theta=0$. Which implies $r_{ij}=s_{ij}=0$ and $G^i_\alpha=0$. So β is parallel with respect to α, and α is flat. In this case, F is isometric to a Minkowski metric $\widetilde{F}(y)=\frac{|y|^2}{|y|-b_iy^i}$, where $|\cdot|$ is the Euclidean metric on $\mathbb{R}^n$, and b_i $(i=1,2,\cdots,n)$ is non-zero constant. Conversely, since β is parallel with respect to α, then $S=0$. Because α is flat , that is $G^i_\alpha=0$, then the left side of equation (5) is equal to zero, so F is locally dually flat. This completes the proof of Theorem 2.

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关于局部对偶平坦且具有迷向 S-曲率的 Matsumoto 度量

蒋 经 农[1,2]

1. 重庆理工大学 数学与统计学院，重庆 400054；2. 遵义医学院 医学信息工程系，贵州 遵义 563003

摘要: 找到了一些方程去刻画局部对偶平坦的 Matsumoto 度量 $F=\frac{\alpha^2}{\alpha-\beta}$，其中 $\alpha=\sqrt{a_{ij}y^iy^j}$，$\beta=b_iy^i$. 同时对局部对偶平坦且具有迷向 S-曲率的 Matsumoto 度量进行了分类.

关键词: 芬斯勒度量；Matsumoto 度量；局部对偶平坦的芬斯勒度量；S-曲率

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