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Bonnesen-Style Inequalities for Non-Simple Closed Plane Curve①

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Abstract: In this paper, some Bonnesen-style isoperimetric inequalities are obtained for a non-simple plane curve via the integral geometry method.

Key words: Bonnesen-style inequality; winding number; circumradius; inradius

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The classical isoperimetric inequality[1] states that: For a simple closed curve C of length L in the plane, the area A enclosed by C satisfies

$$L^2 - 4\pi A \geqslant 0 \tag{1}$$

The equality is attained if and only if C is a circle. It follows that the circle is the only curve of constant length L enclosing the maximunm area.

Bonnesen proved some isoperimetric inequalities that are stronger than inequality (1) (see references [2-3]). These inequalities have come to be known as Bonnesen-style inequalities:

$$L^2 - 4\pi A \geqslant (L - 2\pi r_i)^2$$

$$L^2 - 4\pi A \geqslant (2\pi r_e - L)^2$$

$$L^2 - 4\pi A \geqslant \pi^2 (r_e - r_i)^2$$

where r_e is the circumradius and r_i is the inradius of the curve C, each of the above equalities holds if and only if C encloses a disc.

In general, a Bonnesen-style isoperimetric inequality has the following form:

$$L^2 - 4\pi A \geqslant B$$

where the quantity B on the right-hand side is an invariant of geometric significance having the following basic properties: B is non-negative, and B can vanish only when K is a disc.

Many Bonnesen-style inequalities are found in the last century and mathematicians are still working on those unknown invariants of geometric significance. For the generalizations of those Bonnesen-style inequalities to surfaces of constant curvature, see references [4-10].

The isoperimetric inequality has been obtained by Banchoff and Pohl for a non-simple closed curve (a curve C with self-intersections) in a plane (see references [4,6,8]):

Let C be a curve of length L with self-intersections, the complement of C consists of a number of com-

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ponents D_k and with respect to each domain D_k. C has a well-defined winding number n_k (see references [4-6]), and A_k is the area of the domain D_k (in the usual sense). Then

$$L^2 - 4\pi \sum_k n_k^2 A_k \geqslant 0$$

if and only if C is a circle traversed a finite number of times in a given direction.

One may ask if a non-simple closed curve in a plane has Bonnesen-style inequalities. In this paper, we will answer this question. We use the containment measure of one domain to contain in another domain in a plane and obtain some Bonnesen-type inequalities for a non-simple closed curve in the plane $\mathbb{R}^2$.

Let D_k $(k = i, j)$ be two domains of areas A_k with simple boundaries of circumlengths L_k in the plane $\mathbb{R}^2$. For $g \in G$, the group of isometry in $\mathbb{R}^2$, references [9-10] has the following containment measure inequality

$$\mathrm{m}\{g \in G: gD_i \subset D_j \text{ or } gD_i \supset D_j\} \geqslant 2\pi(A_i + A_j) - L_i L_j \tag{2}$$

If we let $D_i = D$, and let D_j be a disc of radius r between the inscribed disc of radius r_i and the circumscribed disc of radius r_e of D, that is $r_i \leqslant r \leqslant r_e$, we have neither $gD \subset D_j$ nor $gD \supset D_j$ for any $g \in G$. Then $\mathrm{m}\{g \in G: gD \subset D_j \text{ or } gD \supset D_j\} = 0$, and the inequality (2) leads to the following Bonnesen inequality:

Lemma 1[9-10] Let C be a simple closed plane curve of length L bounding an area A, and let r_e be the circumradius and r_i be the inradius of the curve C. Then equality $\pi r^2 - Lr + A \leqslant 0$ $(r_i \leqslant r \leqslant r_e)$ holds if and only if C is a circle.

We will use Lemma 1 to prove our main result.

Theorem 1 Let C be a curve of length L with self-intersections, the complement of C consists of a number of components D_k and with respect to each domain D_k. n_k is the winding number of D_k, L_k is the length of bounding D_k, and A_k is the area (in the usual sense) of the domain D_k. Let R_k be the circumradius and r_k be the inradius of domain D_k. Then

$$L^2 - 4\pi \sum_k n_k^2 A_k \geqslant \sum_k n_k^2 (L_k - 2\pi t_k)^2 \qquad r_k \leqslant t_k \leqslant R_k$$

$$L^2 - 4\pi \sum_k n_k^2 A_k \geqslant \pi^2 \sum_k n_k^2 (R_k - r_k)^2$$

Each equality holds if and only if C is a circle traversed a finite number of times in a given direction.

Proof We may easily know that $L \geqslant \sum_k |n_k| L_k$ if and only if every bounding D_k is traversed $|n_k|$ times in a given direction. By Lemma 1, for every D_k, we have

$$\pi t_k{}^2 - L_k t_k + A_k \leqslant 0 \qquad r_k \leqslant t_k \leqslant R_k$$

We have

$$\pi \sum_k n_k^2 t_k{}^2 - \sum_k n_k^2 L_k t_k + \sum_k n_k^2 A_k \leqslant 0 \qquad r_k \leqslant t_k \leqslant R_k$$

Therefore

$$\begin{aligned}
& L^2 - 4\pi \sum_k n_k^2 A_k \geqslant \\
& \Big(\sum_k |n_k| L_k\Big)^2 - 4\pi \sum_k n_k^2 A_k \geqslant \\
& \sum_k n_k^2 L_k^2 - 4\pi \sum_k n_k^2 L_k t_k + 4\pi^2 \sum_k n_k^2 t_k^2 = \\
& \sum_k n_k^2 (L_k^2 - 4\pi L_k t_k + 4\pi^2 t_k^2) = \\
& \sum_k n_k^2 (L_k - 2\pi t_k)^2
\end{aligned}$$

that is

$$L^2 - 4\pi \sum_k n_k^2 A_k \geqslant \sum_k n_k^2 (L_k - 2\pi t_k)^2 \qquad r_k \leqslant t_k \leqslant R_k$$

Especially, when $t_k = r_k$ or $t_k = R_k$, for every k, we have inequalities:

$$L^2 - 4\pi \sum_k n_k^2 A_k \geqslant \sum_k n_k^2 (L_k - 2\pi r_k)^2 \tag{3}$$

$$L^2 - 4\pi \sum_k n_k^2 A_k \geqslant \sum_k n_k^2 (2\pi R_k - L_k)^2 \tag{4}$$

By inequalities (3) and (4), and $x^2 + y^2 \geqslant \frac{(x+y)^2}{2}$, we have

$$L^2 - 4\pi \sum_k n_k^2 A_k \geqslant \pi^2 \sum_k n_k^2 (R_k - r_k)^2$$

Corollary 1 Let C be a curve of length L with self-intersections, the complement of C consists of a number of components D_k and with respect to each domain D_k, n_k is the winding number of D_k, and A_k is the area of the domain D_k (in the usual sense). Then

$$L^2 - 4\pi \sum_k n_k^2 A_k \geqslant 0$$

if and only if C is a circle traversed a finite number of times in a given direction.

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平面上非简单闭曲线的 Bonnesen 型不等式

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摘要：用积分几何的方法，得到了一些非简单的平面闭曲线的 Bonnesen 型等周不等式.

关键词： Bonnesen 型不等式；环绕数；最小外接圆半径；最大内接圆半径

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