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The Feller Property for the Monotone q-Matrix①

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Abstract: In this paper, a necessary and sufficient condition for a monotone q-matrix $\boldsymbol{Q}$ to be Feller is given in terms of $\widetilde{\boldsymbol{Q}}$, where $\widetilde{\boldsymbol{Q}}$ is the dual of $\boldsymbol{Q}$. Then, the authors further point out that the minimal $\widetilde{\boldsymbol{Q}}$-function $\widetilde{\boldsymbol{P}}(t)$ is the dual for the minimal $\boldsymbol{Q}$-function $\boldsymbol{P}(t)$ if $\boldsymbol{Q}$ is monotone and zero-exit.

Key words: continuous-time Markov chain; transition function; q-matrix; zero-exit; zero-entrance; monotone

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In this paper, we discuss the Feller property for the monotone q-matrix. Feller processes is a basic and important class of continuous-time Markov chains(CTMC), see references[1-2]. In recent years, it is further developed in many works[3-5].

For simplicity, in this paper we just consider CTMC on a linear ordering set. Namely, we assume that the state space E is the natural ordering set, that is, $E=\{0, 1, 2, \cdots\}$.

Recall that a transition function $\boldsymbol{P}(t)=(p_{ij}(t);\ i,j\in E)$ is called to be Feller if

$$p_{ij}(t)\to 0 \qquad i\to\infty,\ t\geqslant 0$$

An infinite matrix $\boldsymbol{Q}=(q_{ij};\ i,j\in E)$ is called to be a q-matrix if

$$\begin{cases} 0\leqslant q_{ij}<+\infty & i\neq j \\ \sum\limits_{j\neq i} q_{ij}\leqslant -q_{ii}:=q_i\leqslant +\infty \end{cases}$$

we call $\boldsymbol{Q}$ is stable if $q_i<+\infty$ $(i\in E)$. In this paper, we only consider stable q-matrix.

Definition 1[4]　A q-matrix $\boldsymbol{Q}=(q_{ij};\ i,j\in E)$ is called to be dual if

$$\sum_{k=0}^{j} q_{ik}\geqslant \sum_{k=0}^{j} q_{i+1,k} \qquad j\neq i$$

$\boldsymbol{Q}$ is monotone if

$$\sum_{k\geqslant j} q_{ik}\leqslant \sum_{k\geqslant j} q_{i+1,k} \qquad j\neq i+1$$

$\boldsymbol{Q}$ is Feller if

$$q_{ij}\to 0 \qquad i\to\infty$$

Definition 2[4]　A transition function $\boldsymbol{P}(t)=(p_{ij}(t);\ i,j\in E)$ is monotone if $\sum\limits_{j\geqslant k} p_{ij}(t)$ is a non-decreasing function of i for fixed $k\in E$ and $t\geqslant 0$.

Definition 3[4]　A transition function $\boldsymbol{P}(t)=(p_{ij}(t);\ i,j\in E)$ is a dual if $\sum\limits_{k=0}^{j} p_{ik}(t)\downarrow 0$ as $i\to\infty$ for $j\in E$ and $t\geqslant 0$.

Proposition 1[1]　(Siegmund's Theorem) A transition function $\boldsymbol{P}(t)$ is monotone if and only if there

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exists a dual $\widetilde{\boldsymbol{P}}(t)$ for $\boldsymbol{P}(t)$ (namely, if and only if there exists another transition function $\widetilde{\boldsymbol{P}}(t)$) such that

$$\sum_{k=0}^{j}\widetilde{p}_{ik}(t)=\sum_{k=i}^{\infty}p_{jk}(t)\qquad \forall\, i,j\in E,\ t\geqslant 0$$

In fact, a monotone q-function $\boldsymbol{P}(t)$ and its dual $\widetilde{\boldsymbol{P}}(t)$ are totally determined with each other as

$$\widetilde{p}_{ij}(t)=\sum_{k=i}^{\infty}(p_{jk}(t)-p_{j-1,k}(t))\qquad \forall\, i,j\in E,\ t\geqslant 0$$

where $p_{-1,k}(t)=0$ and

$$p_{ij}(t)=\sum_{k=0}^{i}(\widetilde{p}_{jk}(t)-\widetilde{p}_{j+1,k}(t))\qquad \forall\, i,j\in E,\ t\geqslant 0 \tag{1}$$

Proposition 2　Transition function $\widetilde{\boldsymbol{P}}(t)$ is a dual if and only if there exists a monotone transition function $\boldsymbol{P}(t)$ satisfying formula (1).

proof　If transition function $\widetilde{\boldsymbol{P}}(t)$ is a dual, then there exists a monotone transition function $\boldsymbol{P}(t)$ satisfying formula (1), which is an improvement of Proposition 1.2 in reference [3]. Conversely, it follows from Proposition 1 that if there exists a monotone transition function $\boldsymbol{P}(t)$ satisfying formula (1) so $\widetilde{\boldsymbol{P}}(t)$ is a dual.

Proposition 3[4]　Let $\boldsymbol{Q}$ be a monotone q-matrix and define the dual q-matrix $\widetilde{\boldsymbol{Q}}$ for $\boldsymbol{Q}$ by

$$\widetilde{q}_{ij}=\sum_{k=i}^{\infty}(q_{jk}-q_{j-1,k})\qquad i,j\in E \tag{2}$$

where $q_{-1,k}=0$. Then $\widetilde{\boldsymbol{Q}}$ is a Feller q-matrix.

Definition 4[4]　A q-matrix $\boldsymbol{Q}$ is called to be zero-exit if $l_{\infty}^{+}(\lambda)=0$ (or equivalently if $l_{\infty}(\lambda)=0$), $\boldsymbol{Q}$ is zero-entrance if $l_{1}^{+}(\lambda)=0$ and strong zero-entrance if $l_{1}(\lambda)=0$, where

$$l_{\infty}(\lambda)=\{x\in l_{\infty}\mid(\lambda\boldsymbol{I}-\boldsymbol{Q})\mathrm{x}=0\}\qquad \mathrm{l}_{\infty}^{+}(\lambda)=\{\mathrm{x}\in \mathrm{l}_{\infty}(\lambda)\mid \mathrm{x}\geqslant 0\}$$

$$\mathrm{l}_{1}(\lambda)=\{\mathrm{x}\in \mathrm{l}_{1}\mid \mathrm{y}(\lambda\boldsymbol{I}-\boldsymbol{Q})=0\}\qquad \mathrm{l}_{1}^{+}(\lambda)=\{\mathrm{x}\in \mathrm{l}_{1}(\lambda)\mid \mathrm{y}\geqslant 0\}$$

It is well known that $l_{\infty}^{+}(\lambda)=0$ is equivalent to $l_{\infty}(\lambda)=0$, which has been proved by Theorem 2.2.7 of reference [1]. For the question whether zero-entrance is equivalent to strong zero-entrance or not, even through references [4—5] have done some works for several specific cases, however, the question is still open for the general cases.

1　Main results

Theorem 1　Let $\boldsymbol{Q}$ be a monotone q-matrix, and $\widetilde{\boldsymbol{Q}}$ is defined as formula (2), then $\boldsymbol{Q}$ is Feller if and only if $\widetilde{\boldsymbol{Q}}$ is monotone.

proof　Necessity. Since $\boldsymbol{Q}$ is monotone, then $\lim\limits_{j\to\infty}\sum\limits_{k=i}^{\infty}q_{jk}=b_i$ $(i\leqslant j)$. Since $\boldsymbol{Q}$ is Feller, it follows that

$$b_0=\lim_{j\to\infty}\sum_{k=0}^{\infty}q_{jk}=\lim_{j\to\infty}\sum_{k=0}^{i-1}q_{jk}+\lim_{j\to\infty}\sum_{k=i}^{\infty}q_{jk}=b_i\qquad i\leqslant j$$

$$\lim_{j\to\infty}\sum_{k=i}^{\infty}q_{jk}=\lim_{j\to\infty}\sum_{k=0}^{\infty}q_{jk}-\lim_{j\to\infty}\sum_{k=0}^{i-1}q_{jk}=b_0\qquad i>j$$

Thus b_i is independent of $i\in E$. Letting $\widetilde{b}_i=\sum\limits_{k=0}^{\infty}\widetilde{q}_{ik}$ for $i\in E$ and using formula (2) we get

$$\widetilde{b}_i=\lim_{j\to\infty}\sum_{k=0}^{j}\widetilde{q}_{ik}=\lim_{j\to\infty}\sum_{k=i}^{\infty}q_{jk}=b_i=b_0\qquad i\in E$$

so $\widetilde{b}_i$ is independent of $i\in E$. Thus

$$\sum_{k=j}^{\infty}\widetilde{q}_{ik}=\sum_{k=0}^{\infty}\widetilde{q}_{ik}-\sum_{k=0}^{j-1}\widetilde{q}_{ik}=b_0-\sum_{k=0}^{j-1}\widetilde{q}_{ik}$$

with the fact that $\widetilde{\boldsymbol{Q}}$ is dual, that is $\sum\limits_{k=0}^{j-1}\widetilde{q}_{ik}\geqslant\sum\limits_{k=0}^{j-1}\widetilde{q}_{i+1,k}$ $(j\neq i+1)$. Thus, $\sum\limits_{k=j}^{\infty}\widetilde{q}_{ik}\leqslant\sum\limits_{k=j}^{\infty}\widetilde{q}_{i+1,k}$ for $j\neq i+1$, which means that $\widetilde{\boldsymbol{Q}}$ is monotone.

Sufficiency. Since $\widetilde{\boldsymbol{Q}}$ is dual, then $\sum\limits_{j=0}^{k}\widetilde{q}_{ij}\geqslant\sum\limits_{j=0}^{k}\widetilde{q}_{i+1,j}$ for $k>i$. Letting $k\to\infty$, we have $\sum\limits_{j=0}^{\infty}\widetilde{q}_{ij}\geqslant\sum\limits_{j=0}^{\infty}\widetilde{q}_{i+1,j}$, the monotonicity of $\widetilde{\boldsymbol{Q}}$ implies that $\sum_{j=0}^{\infty}\tilde{q}_{ij} \leqslant \sum_{j=0}^{\infty}\tilde{q}_{i+1,j}$.

Thus $\sum_{j=0}^{\infty}\tilde{q}_{ij}=\tilde{b}$, which means that $\sum_{j=0}^{\infty}\tilde{q}_{ij}$ is independent of $i\in E$. Therefore, $\lim_{i\to\infty}\sum_{k=j}^{\infty}q_{ik}=\lim_{i\to\infty}\sum_{k=0}^{i}\tilde{q}_{jk}=\tilde{b}=b$ for $j\in E$. Then $\lim_{i\to\infty}q_{ij}=\lim_{i\to\infty}(\sum_{k=j}^{\infty}q_{ik}-\sum_{k=j+1}^{\infty}q_{ik})=b-b=0$, that is, $\boldsymbol{Q}$ is Feller.

Theorem 2 Let $\widetilde{\boldsymbol{Q}}$ be defined as formula (2) and $\widetilde{\boldsymbol{P}}(t)$ be the minimal $\widetilde{\boldsymbol{Q}}$-function, then $\widetilde{\boldsymbol{P}}(t)$ is the dual for $\boldsymbol{P}(t)$ (the minimal $\boldsymbol{Q}$-function) if $\boldsymbol{Q}$ is monotone and zero-exit.

Proof Since $\boldsymbol{Q}$ is zero-exit, so $\widetilde{\boldsymbol{Q}}$ is strong zero-entrance (see Theorem 3.4 of reference [4]). Thus, together with the fact that $\widetilde{\boldsymbol{Q}}$ is dual and Feller (see Proposition 3), it follows from Theorem 3.2 of reference [4] that transition function $\widetilde{\boldsymbol{P}}(t)$ is a dual. By Proposition 2 we know that there exists a monotone transition function $\boldsymbol{P}^{(1)}(t)$ satisfying

$$\boldsymbol{p}_{ij}^{(1)}(t)=\sum_{k=0}^{i}(\tilde{p}_{jk}(t)-\tilde{p}_{j+1,k}(t)) \qquad \forall\, i,j\in E,\ t\geqslant 0$$

then we easily obtain the following equation

$$q_{ij}^{(1)}=\sum_{k=0}^{i}(\tilde{q}_{jk}-\tilde{q}_{j+1,k}) \qquad \forall\, i,j\in E$$

Thus $q_{ij}^{(1)}=q_{ij}$, which means that $\boldsymbol{P}^{(1)}(t)$ is a $\boldsymbol{Q}$-function. Since $\boldsymbol{Q}$ is zero-exit, we have $\boldsymbol{P}^{(1)}(t)=\boldsymbol{P}(t)$ (see Theorem 2.2.7 of reference [1]). Thus $\widetilde{\boldsymbol{P}}(t)$ is the dual for $\boldsymbol{P}(t)$.

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单调 q 矩阵的 Feller 性质

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摘要：给出了单调 q-矩阵 $\boldsymbol{Q}$ 是 Feller 的充分必要条件，进一步指出：若 q-矩阵 $\boldsymbol{Q}$ 是单调零流出的且 $\widetilde{\boldsymbol{Q}}$ 是 $\boldsymbol{Q}$ 的对偶，则最小 $\widetilde{\boldsymbol{Q}}$-函数 $\widetilde{\boldsymbol{P}}(t)$是最小 $\boldsymbol{Q}$-函数 $\boldsymbol{P}(t)$的对偶.

关键词：连续时间马尔科夫链；转移函数；q 矩阵；零流出；零流入；单调

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