文章编号:1000-5471(2011)02-0024-04

The Asymptotic Behavior of Solutions for a Nonlinear Higher Order Kirchhoff Type Equation①

LI Yan

College of mathematics, Chengdu University of Information Technology, Chengdu 610225, China

Abstract: In this paper, the auther deals with the nonlinear higher order Kirchhoff type equation with weak dissipation

$$u_{tt}+\left(\int_{\Omega}|D^{m}u|^{2}\mathrm{d}x\right)^{q}(-\Delta)^{m}u+\beta u_{t}+g(u)=0 \qquad x\in\Omega,\ t>0$$

in a bounded domain, where $m>1$ is a positive integer, and $q>0$ is a positive constant. The author studies the asymptotic behavior of solutions to this equationin $E_0=H^m(\Omega)\times H(\Omega)$ and proves that the solutions to this equation are of asymptotic stability.

Key words: Kirchhoff type equation; higher order; asymptotic behavior

CLC number: O175.29 **Document code**: A

In this paper, the main purpose is to study the asymptotic behavior of the solutions to a nonlinear higher order Kirchhoff type equation. To formalize this problem, we take Ω be an open bounded set of $\mathbb{R}^n (n\geqslant 1)$ with smooth boundary Γ. Let us denote by $\upsilon(x)$ the unit normal vector at $x\in\Gamma$ outside of Ω and let us consider the following initial boundary value problem:

$$u_{tt}+\left(\int_{\Omega}|D^{m}u|^{2}\mathrm{d}x\right)^{q}(-\Delta)^{m}u+\beta u_{t}+g(u)=0,\ \text{in } Q=\Omega\times(0,+\infty) \tag{1}$$

$$u(x,t)=0,\ \frac{\partial^{i}u}{\partial\upsilon^{i}}=0,\ i=1,2,\cdots,m-1,\ \text{on } \sum=\Gamma\times(0,+\infty) \tag{2}$$

$$u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ \text{in } x\in\Omega \tag{3}$$

where u is the transverse displacement and the function $g\in C^1$ satisfies the following conditions:

$$\liminf_{|s|\to\infty}\frac{G(s)}{s^2}\geqslant 0,\ G(s)=\int_0^s g(r)\mathrm{d}r \tag{4}$$

$$\limsup_{|s|\to\infty}\frac{|g'(s)|}{|s|^{\gamma}}=0 \tag{5}$$

where $0\leqslant\gamma<+\infty$ $(n=1, 2)$, $0\leqslant\gamma<2(n=3)$, $\gamma=0(n\geqslant 4)$. Furthermore, there exists $C_1>0$ such that

$$\liminf_{|s|\to\infty}\frac{sg(s)-C_1G(s)}{s^2}\geqslant 0 \tag{6}$$

For example, when $g(u)=|u|^{\gamma}u$, then (4)-(6) hold with $C_1=1+\gamma$. Problem (1)-(3) has its origin from the mathematical description of small amplitude vibrations of an elastic string[1]. In fact, a mathematical model deflecting an elastic string of length $L>0$ is given by the mixed problem for the nonlinear wave equation

---

① 收稿日期:2009-11-16
基金项目:四川省教育厅重点项目(09ZA091);成都信息工程学院基金项目(CRF201006);成都信息工程学院引进人才科研启动基金(KYTZ201004).
作者简介:李 艳(1976-),女,四川自贡人,讲师,主要从事应用泛函分析研究.

$$\rho h \frac{\partial^2 u}{\partial t^2} = \left\{ p_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{d}x \right\} \frac{\partial^2 u}{\partial x^2}, \text{ for } 0 < x < L,\ t \geqslant 0 \tag{7}$$

where u is the lateral deflection, x is the space coordinate, t is the time coordinate, E is the Young's modulus, ρ is the mass density, h is the cross section area and p_0 is the initial axial tension. Kirchhoff was the first one to introduce (7) in the study of oscillations of stretched strings and plates, so that (7) is named after Kirchhoff as the wave equation of Kirchhoff type. The existence of global solutions and exponential decay to Kirchhoff equation (1) have been investigated by many authors[2-6]. Especially, when $m=q=1$, $g(u)=|u|^{\gamma}u$, and $0\leqslant\gamma<\frac{2}{n-2}(n\geqslant3)$, Papadopoulos and Stavrakakis[6] have proved the global existence of solutions and blow-up results for (1) on R^n. There are some authors who have studied the solutions of nonlinear elliptic equation[7] and nonlinear parabolic equation[8-9].

Throughout this paper we define $H(\Omega)=L^2(\Omega)$, $(u,v):=\int_\Omega u(x)v(x)\mathrm{d}x$, $\forall u,v\in L^2(\Omega)$, and $\| u \| = \| u \|_{L^2(\Omega)}$. We also set $E_0=H^m(\Omega)\times H(\Omega)$.

we will study the asymptotic behavior of solutions for problem (1)-(3). The main result of the paper is as follows.

Theorem 1 Assume that $(u_0, u_1)\in E_0$, then the problem (1)-(3) admits a unique global solution u satisfying

$$u \in C(0,+\infty;\ H^m(\Omega)),\ u_t \in C(0,+\infty;\ H(\Omega))$$

Moreover, there exists a $t_0>0$ and a constant $\rho_0>0$ such that

$$\| D^m u(t) \|^2 + \| u_t(t) \|^2 \leqslant \rho_0^2, \text{ for } t \geqslant t_0 \tag{8}$$

Proof We apply the methods of Galerkin approximation to equation(1) of which the key step is priori estimates. By (4),(6), and apply the Poincaré inequality, there exist constants $K_1, K_2>0$ only depending on u, such that

$$\int_\Omega G(u(x))\mathrm{d}x + \eta \| D^m u \|^2 + K_1 \geqslant 0,\ \forall u \in H^m \tag{9}$$

$$\int_\Omega ug(u)\mathrm{d}x - C_1\int_\Omega G(u(x))\mathrm{d}x + \eta \| D^m u \|^2 + K_2 \geqslant 0,\ \forall u \in H^m \tag{10}$$

We take the scalar product in H of equation (1) with $\tilde{u}=u_t+\alpha u$, $0<\alpha\leqslant\alpha_0$, α_0 will be chosen later. After a computation, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\| \tilde{u} \|^2 + \frac{1}{q+1} \| D^m u \|^{2q+2} + 2\int_\Omega G(u)\mathrm{d}x \right) + \\ &\alpha \| D^m u \|^{2q+2} + (\beta-\alpha) \| \tilde{u} \|^2 - \\ &\alpha(\beta-\alpha)\int_\Omega u\tilde{u}\,\mathrm{d}x + \alpha\int_\Omega g(u)u\mathrm{d}x = 0 \end{aligned} \tag{11}$$

We note that

$$\begin{aligned} \alpha\int_\Omega g(u)u\mathrm{d}x \geqslant &\alpha C_1\int_\Omega G(u)\mathrm{d}x - \frac{\alpha}{4} \| D^m u \|^2 - \alpha K_2 \geqslant \\ &\alpha C_1\int_\Omega G(u)\mathrm{d}x - \frac{\alpha}{4} \| D^m u \|^{2q+2} - C(q) - \alpha K_2 \end{aligned}$$

where $C(q)$ is a constant. On the other hand, we have

$$\begin{aligned} &(\beta-\alpha) \| \tilde{u} \|^2 - \alpha(\beta-\alpha)\int_\Omega u\tilde{u}\,\mathrm{d}x \geqslant \\ &(\beta-\alpha) \| \tilde{u} \|^2 - \frac{\alpha(\beta-\alpha)}{\sqrt{\lambda_1}} \| D^m u \| \, \| \tilde{u} \| \geqslant \\ &(\beta-\alpha) \| \tilde{u} \|^2 - \frac{\alpha}{4} \| D^m u \|^2 - \frac{\alpha(\beta-\alpha)^2}{\lambda_1} \| \tilde{u} \|^2 \geqslant \\ &\left(\beta-\alpha-\frac{\alpha(\beta-\alpha)^2}{\lambda_1}\right) \| \tilde{u} \|^2 - \frac{\alpha}{4} \| D^m u \|^2 \geqslant \end{aligned}$$

$$\left(\beta-\alpha-\frac{\alpha(\beta-\alpha)^2}{\lambda_1}\right)\|\tilde{u}\|^2-\frac{\alpha}{4}\|D^m u\|^{2q+2}-C(q)$$

where λ_1 is a constant, $\lambda_1=\inf\limits_{v\in H^m,\ v\neq 0}\frac{\|\nabla^m v\|^2}{\|v\|^2}$. Thus we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\tilde{u}\|^2+\frac{1}{q+1}\|D^m u\|^{2q+2}+2\int_\Omega G(u)\mathrm{d}x\right)+$$
$$\left(\beta-\alpha-\frac{\alpha(\beta-\alpha)^2}{\lambda_1}\right)\|\tilde{u}\|^2+\frac{\alpha}{2}\|D^m u\|^{2q+2}+$$
$$\alpha C_1\int_\Omega G(u)\mathrm{d}x\leqslant\alpha K_2+2C(q) \tag{12}$$

Choose α_0 such that

$$\alpha_0\left(1+\frac{(\beta-\alpha)^2}{\lambda_1}\right)=\frac{\beta}{2}$$

Since $1+\frac{(\beta-\alpha)^2}{\lambda_1}>1$, and $0<\alpha\leqslant\alpha_0$, we have

$$\beta-\alpha-\frac{\alpha(\beta-\alpha)^2}{\lambda_1}\geqslant\frac{\beta}{2}\geqslant\alpha_0\geqslant\alpha$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\tilde{u}\|^2+\frac{1}{q+1}\|D^m u\|^{2q+2}+2\int_\Omega G(u)\mathrm{d}x\right)+$$
$$\alpha\left(\|\tilde{u}\|^2+\|D^m u\|^{2q+2}+2C_1\int_\Omega G(u)\mathrm{d}x\right)\leqslant 2\alpha K_2+4C(q)$$

Set $\delta=\min(\alpha,\alpha C_1)$ and then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\tilde{u}\|^2+\frac{1}{q+1}\|D^m u\|^{2q+2}+2\int_\Omega G(u)\mathrm{d}x+2K_1\right)+$$
$$\delta\left(\|\tilde{u}\|^2+\frac{1}{q+1}\|D^m u\|^{2q+2}+2\int_\Omega G(u)\mathrm{d}x+2K_1\right)\leqslant$$
$$2\alpha K_2+4C(q)+2\delta K_1 \tag{13}$$

Set

$$y(t)=\|\tilde{u}\|^2+\frac{1}{q+1}\|D^m u\|^{2q+2}+2\int_\Omega G(u)\mathrm{d}x+2K_1$$

From condition (9) we know $y(t)\geqslant 0$ and (13) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t)+\delta y(t)\leqslant 2\alpha K_2+4C(q)+2\delta K_1$$

By Gronwall's inequality, we obtain

$$y(t)\leqslant y(0)\mathrm{e}^{-\delta t}+\frac{2\alpha K_2+4C(q)+2\delta K_1}{\delta}(1-\mathrm{e}^{-\delta t}),\ t\geqslant 0 \tag{14}$$

For any bounded subset B of E_0, $(u_0,u_1)\in B$, $\|D^m u_0\|^{2q+2}$ and $\int_\Omega G(u_0)dx$ are bounded, too. Hence

$$R=R(B)=\sup_{(u_0,u_1)\in B}y(0)=$$
$$\sup_{(u_0,u_1)\in B}\left\{\|u_1+\alpha u_0\|^2+\|D^m u_0\|^2+2\int_\Omega G(u_0)\mathrm{d}x+2K_1\right\}<+\infty$$

and

$$\lim_{t\to\infty}\sup_{(u_0,u_1)\in B}y(t)\leqslant\frac{2\alpha K_2+4C(q)+2\delta K_1}{\delta}\equiv\mu_0^2 \tag{15}$$

Let $\mu_1>\mu_0$ be fixed, and

$$t_0=t_0(R,\mu_1)=\frac{1}{\alpha}\ln\frac{R}{\mu_1^2-\mu_0^2}$$

for any $t\geqslant t_0$, we have $y(t)\leqslant\mu_1^2$ and

$$\|D^m u(t)\|^2+\|u_t(t)\|^2=$$

$$
\begin{aligned}
&\| D^m u(t) \|^2 + \| u_t(t) + \alpha u(t) - \alpha u(t) \|^2 \leqslant \\
&\| D^m u(t) \|^2 + 2(\| u_t(t) + \alpha u(t) \|^2 + \alpha^2 \| u(t) \|^2) \leqslant \\
&\| D^m u(t) \|^2 + 2 \| u_t(t) + \alpha u(t) \|^2 + \frac{2\alpha^2}{\lambda_1} \| D^m u(t) \|^2 \leqslant \\
&\left(1 + \frac{2\alpha^2}{\lambda_1}\right) \| D^m u(t) \|^2 + 2 \| u_t(t) + \alpha u(t) \|^2 \leqslant \\
&\max\left\{1 + \frac{2\alpha^2}{\lambda_1}, 2\right\} (\| D^m u(t) \|^2 + \| u_t(t) + \alpha u(t) \|^2) \leqslant \\
&\max\left\{1 + \frac{2\alpha^2}{\lambda_1}, 2\right\} y(t) \leqslant \\
&\max\left\{1 + \frac{2\alpha^2}{\lambda_1}, 2\right\} \mu_1^2
\end{aligned}
$$

Thus we obtain

$$
\| D^m u(t) \|^2 + \| u_t(t) \|^2 \leqslant \rho_0^2, \text{ for } t \geqslant t_0
$$

where $\rho_0^2 = \max\left\{1 + \frac{2\alpha^2}{\lambda_1}, 2\right\} \mu_1^2$. The proof of Theorem 1 is completed.

References：

[1] NARASIMHA K. Nonlinear Vibration of an Elastic String [J]. J Sound Vibration , 1968(8)：134－146.

[2] BILER P. Remark on the Decay for Damped String and Beam Equations [J]. Nonlinear Anal TMA，1984，9(1)：839－842.

[3] BRITO E H. Nonlinear Initial Boundary Value Problems [J]. Nonlinear Anal TMA ，1987，11(1)：125－137.

[4] MATSUYAMA T，IKERATA R. On Global Solutions and Energy Decay for the Wave Equations of Kirchhoff Type with Nonlinear Damping Terms [J]. J Math Anal Appl，1996，204，729－753.

[5] NISHIHARA K. Exponentially Decay of Solutions of Some Quasilinear Hyperbolic Equations with Linear Damping [J]. Nonlinear Anal TMA，1984，8(6)：623－636.

[6] PAPADOPOULOS P G，STAVRAKAKIS N M. Global Existence and Blow-up Results for an Equation of Kirchhoff Type on R^N[J]. Topological Methods in Nonlinear Analysis，2001(17)：91－109.

[7] 欧增奇，唐春雷. 一类半线性椭圆方程解的存在性 [J]. 西南师范大学学报：自然科学版，2007，32(1)：1－5.

[8] 徐　思，周　军，穆春来. 一类具有非线性耦合边界条件的非线性扩散系统的爆破分析 [J]. 西南师范大学学报：自然科学版，2007，32(3)：10－14.

[9] 李中平，王雄瑞. 一类带有局部化源的反应扩散方程组解的整体存在性及爆破 [J]. 西南师范大学学报：自然科学版，2007，32(3)：15－18.

非线性高阶 Kirchhoff 型方程解的渐近性

李　艳

成都信息工程学院 数学学院，成都 610225

摘要：本文考虑有界区域上弱耗散非线性高阶 Kirchhoff 型方程

$$
u_{tt} + \left(\int_\Omega | D^m u |^2 \mathrm{d}x\right)^q (-\Delta)^m u + \beta u_t + g(u) = 0 \qquad x \in \Omega, t > 0
$$

这里 $m>1$ 为正整数，$q>0$ 为正常数. 研究了该方程在 $E_0 = H^m(\Omega) \times H(\Omega)$ 中解的渐近性态，证明了该方程的解是渐近稳定的.

关键词： Kirchhoff 型方程；高阶；渐近性

责任编辑　张　枸