文章编号:1000-5471(2011)01-0073-06

Some Classes of Singular Integral Equations and Riemann Boundary Value Problems with a Parametric Unknown Function①

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Abstract: Some classes of Riemann boundary value problems with a parametric unknown function in multiply connected domains are proposed, and their solutions are obtained. Then using the above results, a method for solving certain singular integral equations with a parametric unknown function is given, and the solution of the above singular integral equations is obtained.

Key words: singular integral equations; Riemann boundary value problem; parametric unknown function; multiply connected domains

CLC number: O175.8 **Document code**: A

It is known to all that singular integral equations with kernels of Cauchy type and Riemann boundary value problems are very important problems with many applications to everyday problems of science and engineering. Singular integral equations with kernels of Cauchy type and Riemann boundary value problems have been investigated completely[1-2]. In reference [3], singular integral equations with a transformation were discussed. In references [4-5], on the basis of mechanical background, some classes of Riemann boundary value problems with a parametric unknown function were presented and investigated preliminarily. In reference [6], a class of Riemann boundary value problems with a parametric unknown function presented by LU Jian-ke was discussed. In reference [7], the Riemann boundary value problems with parametric unknown functions on infinite straight line were investigated.

In this paper, singular integral equations with a parametric unknown function are mainly considered. Firstly, we propose a mathematical formulation of the general Riemann boundary value problems with a parametric unknown function in multiply connected domains, and obtain the solutions of the normal case, which include the result in reference [5] as a special case. Secondly, we transfer the above singular integral equations with a parametric unknown function into the Riemann boundary value problems with a parametric unknown function. Finally, we obtain the solutions of the singular integral equations with a parametric unknown function.

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① 收稿日期:2009-08-10
基金项目:重庆市教育委员会科学技术研究基金资助项目(KJ051206).
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1 Riemann Boundary Value Problems with a Parametric Unknown Function

Let S^+ be a multiply connected domain, whose boundaries are smooth closed curves $\{L_0, L_1, \cdots, L_N\}$. Let L denote $\{L_0, L_1, \cdots, L_N\}$. Suppose $L_1, L_2, \cdots, L_N$ are in the interior of L_0. $S_1, S_2, \cdots, S_N$ denote the bounded domains bounded by $L_1, L_2, \cdots, L_N$ respectively. S^- denotes the remainder set of $S^+ + L$ in the complex plane. Without loss of generality, we can assume that the origin belongs to S^+.

In this paper, we discuss Riemann boundary value problems with a parametric unknown function in the multiply connected domain, which will be called problem R^{-1}. Our problem is to find a pair of functions $(\Phi(z), \psi(t))$, where $\Phi(z)$ is a sectionally holomorphic function with jumps on L, and $\psi(t)$ is a Hölder continuous function on L, satisfying the following boundary conditions

$$\begin{aligned} G_{11}(t)\Phi^+(t) &= G_{12}(t)\Phi^-(t) + c_1(t)\psi(t) + f_1(t) \\ G_{21}(t)\Phi^+(t) &= G_{22}(t)\Phi^-(t) + c_2(t)\psi(t) + f_2(t) \end{aligned} \qquad t \in L \tag{1}$$

where $G_{jk}(t)$, $c_j(t)$, $f_j(t)(j, k=1,2)$ are the given Hölder continuous functions on L. For definiteness, $\Phi(\infty)=0$ is required. We say that Problem R^{-1} is normal if $G_j(t)\neq 0$, $j=1,2$, and κ is termed the index of problem R^{-1}; otherwise, we say that problem R^{-1} is nonnormal, where

$$G_1(t) = \begin{vmatrix} G_{11}(t) & c_1(t) \\ G_{21}(t) & c_2(t) \end{vmatrix}, \ G_2(t) = \begin{vmatrix} G_{12}(t) & c_1(t) \\ G_{22}(t) & c_2(t) \end{vmatrix} \tag{2}$$

$$\kappa = \frac{\left[\lg \dfrac{G_2(t)}{G_1(t)}\right]_L}{2\pi i} = \frac{\left[\arg \dfrac{G_2(t)}{G_1(t)}\right]_L}{2\pi} \tag{3}$$

problem R^{-1} is homogeneous if $f_1(t)=f_2(t)=0$.

The homogeneous and nonhomogeneous problem R^{-1} will be treated separately.

1.1 Homogeneous Problem R^{-1}

The homogeneous problem R^{-1} possesses the following boundary conditions

$$\begin{aligned} G_{11}(t)\Phi^+(t) &= G_{12}(t)\Phi^-(t) + c_1(t)\psi(t) \\ G_{21}(t)\Phi^+(t) &= G_{22}(t)\Phi^-(t) + c_2(t)\psi(t) \end{aligned} \qquad t \in L \tag{4}$$

Multiplying the first and the second conditions of (4) by $c_2(t)$ and $c_1(t)$ respectively, we get

$$G_1(t)\Phi^+(t) = G_2(t)\Phi^-(t) \qquad t \in L \tag{5}$$

Due to $G_j(t)\neq 0$, $j=1,2$, (5) can be written as

$$\Phi^+(t) = G(t)\Phi^-(t) \qquad t \in L \tag{6}$$

where

$$G(t) = \frac{G_2(t)}{G_1(t)} \tag{7}$$

Obviously, $G(t)$ is still the Hölder continuous function, and $G(t)\neq 0$.

The system (6) is a homogeneous Riemann boundary value problem with $\Phi(\infty)=0$. According to references [1-2], the general solution of the system (6) can be represented as

$$\Phi(z) = P_{\kappa-1}(z)X(z) \tag{8}$$

where $P_{\kappa-1}(z)$ is an arbitrary polynomial of degree $\kappa-1$ with $P_{\kappa-1}(z)\equiv 0$ if $\kappa\leqslant 0$, and $X(z)$ as the canonical function of the system (6) possesses the following form

$$X(z) = \begin{cases} X^+(z) = \dfrac{e^{\Gamma(z)}}{\prod(z)} & z \in S^+ \\ X^-(z) = z^{-\kappa} e^{\Gamma(z)} & z \in S^- \end{cases} \tag{9}$$

$$\Gamma(z) = \frac{1}{2\pi i}\int_L \frac{\lg\left[\tau^{-\kappa}\prod(\tau)G(\tau)\right]}{\tau - z} d\tau \qquad \prod(z) = \prod_{j=1}^{N}(z-z_j)^{\kappa_j} \tag{10}$$

in which $z_j \in S_j$, $j=1,2,\cdots,N$, and κ_j is as follows

$$\kappa_j = \frac{[\lg G(t)]_{L_j}}{2\pi i} = \frac{[\arg G(t)]_{L_j}}{2\pi} \qquad j=1,2,\cdots,N \tag{11}$$

Now, substituting the solution $\Phi(z)$ as (8) of the system (6) into (4), and multiplying the first and the second conditions of (4) by $G_{21}(t)$ and $G_{11}(t)$ respectively, we get

$$\psi(t) = D_1(t) P_{\kappa-1}(t) X^-(t) \tag{12}$$

where

$$D_1(t) = -\frac{1}{G_1(t)} \begin{vmatrix} G_{11}(t) & G_{12}(t) \\ G_{21}(t) & G_{22}(t) \end{vmatrix} \tag{13}$$

Finally we get theorem as follows.

Theorem 1 The homogeneous problem R^{-1} possesses κ linearly independent solutions

$$(z^k X(z),\ D_1(t) t^k X^-(t)) \qquad k=0,1,\cdots,\kappa-1$$

and its general solution is $(\Phi(z), \psi(t))$, where $\Phi(z)$ and $\psi(t)$ are given by (8) and (12) respectively when the index $\kappa>0$. When the index $\kappa\leqslant 0$, homogeneous problem R^{-1} only possesses zero-solution $(\Phi(z), \psi(t))=(0, 0)$.

1.2 Nonhomogeneous Problem R^{-1}

Now, we consider the nonhomogeneous problem R^{-1} with the boundary conditions (1). Multiplying the first and the second conditions of (1) by $c_2(t)$ and $c_1(t)$ respectively, we can get

$$\Phi^+(t) = G(t)\Phi^-(t) + f(t),\ t\in L \tag{14}$$

where

$$f(t) = \frac{1}{G_1(t)} \begin{vmatrix} f_1(t) & c_1(t) \\ f_2(t) & c_2(t) \end{vmatrix} \tag{15}$$

Obviously, $f(t)$ is still the Hölder continuous function. According to references [1-2], the system (14) is surely solvable when the index $\kappa\geqslant 0$, and it possesses a particular solution as follows

$$\Phi_0(z) = \frac{X(z)}{2\pi i}\int_L \frac{f(\tau)}{X^+(\tau)(\tau-z)}\mathrm{d}\tau \tag{16}$$

When the index $\kappa<0$, the system (1) is solvable if and only if the following $-\kappa$ conditions hold

$$\int_L \frac{f(\tau)\tau^k}{X^+(\tau)}\mathrm{d}\tau = 0 \qquad k=0,1,\cdots,-\kappa-1 \tag{17}$$

Under the above conditions, the system (14) has a unique solution $\Phi_0(z)$ as (16).

Substituting the particular solution $\Phi_0(z)$ as (16) of the system (14) into (1), and multiplying the first and the second conditions of (1) by $G_{21}(t)$ and $G_{11}(t)$ respectively, we get

$$\psi_0(t) = D_1(t)\Phi_0^-(t) + D_2(t) \tag{18}$$

where

$$D_2(t) = \frac{1}{G_1(t)} \begin{vmatrix} f_1(t) & G_{11}(t) \\ f_2(t) & G_{21}(t) \end{vmatrix} \tag{19}$$

Now, we have obtained the particular solution $(\Phi_0(z), \psi_0(t))$ of the nonhomogeneous problem R^{-1}. On the basis of the general solution of the homogeneous problem R^{-1}, we get the following theorem about the nonhomogeneous problem R^{-1}.

Theorem 2 The nonhomogeneous problem R^{-1} is surely solvable when the index $\kappa\geqslant 0$, and its general solution is $(\Phi(z)+\Phi_0(z), \psi(t)+\psi_0(t))$, where $\Phi(z)$, $\psi(t)$, $\Phi_0(z)$ and $\psi_0(t)$ are given by (8), (12), (16) and (18) respectively, in which $P_{-1}(z)\equiv 0$. When the index $\kappa<0$, the nonhomogeneous problem R^{-1} is solvable if and only if $-\kappa$ conditions given by (17) are satisfied. Under the above conditions, it has a unique solution $(\Phi_0(z), \psi_0(t))$, where $\Phi_0(z)$ and $\psi_0(t)$ are still given by (16) and (18) respectively. In a word, the general solution of the nonhomogeneous problem R^{-1} has κ degree of freedom.

Remark We do not require that $G_{11}(t)\neq 0$ and $G_{21}(t)\neq 0$ in the boundary conditions of The Riemann boundary value problems with a parametric unknown function. This is different from the boundary conditions of classical Riemann boundary value problems obviously.

2 Singular Integral Equations with a Parametric Unknown Function

Using the above results, we may solve the following singular integral equations with a parametric unknown function

$$\begin{aligned} a_1(t)\varphi(t)+\frac{b_1(t)}{\pi i}\int_L\frac{\varphi(\tau)}{\tau-t}\mathrm{d}\tau=c_1(t)\psi(t)+f_1(t) \\ a_2(t)\varphi(t)+\frac{b_2(t)}{\pi i}\int_L\frac{\varphi(\tau)}{\tau-t}\mathrm{d}\tau=c_2(t)\psi(t)+f_2(t) \end{aligned} \qquad t\in L \tag{20}$$

where $a_j(t)$, $b_j(t)$, $c_j(t)$, $f_j(t)(j=1,2)\in H$ are given on L, and satisfying the normal conditions $G_1(t)\neq 0$ and $G_2(t)\neq 0$, where

$$G_1(t)=\begin{vmatrix} a_1(t)+b_1(t) & c_1(t) \\ a_2(t)+b_2(t) & c_2(t)\end{vmatrix},\ G_2(t)=\begin{vmatrix} a_1(t)-b_1(t) & c_1(t) \\ a_2(t)-b_2(t) & c_2(t)\end{vmatrix} \tag{21}$$

We require that the unknown function $\varphi(t)$ and the parametric unknown function $\psi(t)$ be Hölder continuous functions on L.

Now, assume that the equations (20) possesses the solution $(\varphi(t),\ \psi(t))$ with $\varphi(t)$, $\psi(t)\in H$. We introduce the sectionally holomorphic function

$$\Phi(z)=\frac{1}{2\pi i}\int_L\frac{\varphi(\tau)}{\tau-z}\mathrm{d}\tau \qquad z\notin L \tag{22}$$

Then $\Phi(\infty)=0$. Using the Plemelj's formulation of Cauchy type integral

$$\begin{aligned} \varphi(t)=\Phi^+(t)-\Phi^-(t) \\ \frac{1}{\pi i}\int_L\frac{\varphi(\tau)}{\tau-t}d\tau=\Phi^+(t)+\Phi^-(t) \end{aligned} \qquad t\in L \tag{23}$$

the equations (20) is transferred to the following Riemann boundary value problem with a parametric unknown function

$$\begin{aligned} (a_1(t)+b_1(t))\Phi^+(t)=(a_1(t)-b_1(t))\Phi^-(t)+c_1(t)\psi(t)+f_1(t) \\ (a_2(t)+b_2(t))\Phi^+(t)=(a_2(t)-b_2(t))\Phi^-(t)+c_2(t)\psi(t)+f_2(t) \end{aligned} \qquad t\in L \tag{24}$$

From this, we know that $(\Phi(z),\ \psi(t))$ will be the solution of problem R^{-1}(24) if the equations (20) possesses the solution $(\varphi(t),\ \psi(t))$, where $\Phi(z)$ is given in (22).

Conversely, assume that $(\Phi(z),\ \psi(t))$ is the solution of problem R^{-1}(24) with $\Phi(\infty)=0$, and $\varphi(t)=\Phi^+(t)-\Phi^-(t)$. Due to $\varphi(t)=\Phi^+(t)-\Phi^-(t)$ and $\Phi(\infty)=0$, $\Phi(z)$ possesses the form given in (22)[1-2]. Furthermore, $\Phi(z)$ satisfies the following boundary condition

$$\frac{1}{\pi i}\int_L\frac{\varphi(\tau)}{\tau-t}\mathrm{d}\tau=\Phi^+(t)+\Phi^-(t) \tag{25}$$

Now, substituting (25), $\varphi(t)=\Phi^+(t)-\Phi^-(t)$ and $\psi(t)$ into (24), we get the equations (20). Therefore, $(\varphi(t),\ \psi(t))$ is the solution of the equations (20). So the equations (20) is solvable if and only if problem R^{-1}(24) is solvable, and

$$\varphi(t)=\Phi^+(t)-\Phi^-(t),\ \psi(t)=\psi(t) \tag{26}$$

We still say that κ given in (3) with $G_1(t)$, $G_2(t)$ given by (21) is the index of the singular integral equations (20) with a parametric unknown function.

Using the previous notation with

$$G_{11}(t)=a_1(t)+b_1(t),\ G_{12}(t)=a_1(t)-b_1(t)$$
$$G_{21}(t)=a_2(t)+b_2(t),\ G_{22}(t)=a_2(t)-b_2(t) \tag{27}$$

according to Theorem 2, if the index $\kappa\geqslant 0$, then problem $R^{-1}(24)$ possess the general solutions $(\Phi(z),\psi(t))$ as follows

$$\Phi(z)=P_{\kappa-1}(z)X(z)+\frac{X(z)}{2\pi i}\int_L\frac{f(\tau)}{X^+(\tau)(\tau-z)}\mathrm{d}\tau \tag{28}$$

$$\psi(t)=D_1(t)(P_{\kappa-1}(t)X^-(t)+\Phi_0^-(t))+D_2(t) \tag{29}$$

in which $\Phi_0(z)$ is given in (16). If the index $\kappa<0$, then problem $R^{-1}(24)$ is solvable if and only if $-\kappa$ conditions given by (17) are satisfied. Under the above conditions, it has a unique solution $(\varphi(t),\psi(t))$ as (28), (29), in which $P_{\kappa-1}(z)\equiv 0$. From this, we may write out the concrete expression of the solution $(\varphi(t),\psi(t))$ of the equations (20), in which $\varphi(t)=\Phi^+(t)-\Phi^-(t)$. According to (28), (29) and (16), Using the Plemelj's formulation of Cauchy type integral, we can get

$$\varphi(t)=P_{\kappa-1}(t)[X^+(t)-X^-(t)]+\frac{X^+(t)+X^-(t)}{2X^+(t)}f(t)+\frac{X^+(t)-X^-(t)}{2\pi i}\int_L\frac{f(\tau)}{X^+(\tau)(\tau-t)}\mathrm{d}\tau \tag{30}$$

$$\psi(t)=D_1(t)P_{\kappa-1}(t)X^-(t)-\frac{D_1(t)X^-(t)}{2X^+(t)}f(t)+\frac{D_1(t)X^-(t)}{2\pi i}\int_L\frac{f(\tau)}{X^+(\tau)(\tau-t)}\mathrm{d}\tau+D_2(t) \tag{31}$$

Due to the canonical function $X(z)$ satisfying the following condition[1-2]

$$\frac{X^+(t)}{X^-(t)}=G(t)=\frac{G_2(t)}{G_1(t)} \tag{32}$$

(30) and (31) can be written as follows

$$\varphi(t)=\frac{G_2(t)-G_1(t)}{G_2(t)}P_{\kappa-1}(t)X^+(t)+\frac{G_2(t)+G_1(t)}{2G_2(t)}f(t)+\frac{G_2(t)-G_1(t)}{2\pi iG_2(t)}X^+(t)\int_L\frac{f(\tau)}{X^+(\tau)(\tau-t)}\mathrm{d}\tau \tag{33}$$

$$\psi(t)=\frac{D_1(t)G_1(t)}{G_2(t)}P_{\kappa-1}(t)X^+(t)+D_2(t)-\frac{D_1(t)G_1(t)}{2G_2(t)}f(t)+\frac{D_1(t)G_1(t)}{2\pi iG_2(t)}X^+(t)\int_L\frac{f(\tau)}{X^+(\tau)(\tau-t)}\mathrm{d}\tau \tag{34}$$

or

$$\varphi(t)=\frac{G_2(t)-G_1(t)}{G_2(t)}P_{\kappa-1}(t)X^+(t)+Kf \tag{35}$$

$$\psi(t)=\frac{D_1(t)G_1(t)}{G_2(t)}P_{\kappa-1}(t)X^+(t)+D_2(t)+K^*f \tag{36}$$

in which

$$Kf=\frac{G_2(t)+G_1(t)}{2G_2(t)}f(t)+\frac{G_2(t)-G_1(t)}{2\pi iG_2(t)}X^+(t)\int_L\frac{f(\tau)}{X^+(\tau)(\tau-t)}\mathrm{d}\tau \tag{37}$$

$$K^*f=-\frac{D_1(t)G_1(t)}{2G_2(t)}f(t)+\frac{D_1(t)G_1(t)}{2\pi iG_2(t)}X^+(t)\int_L\frac{f(\tau)}{X^+(\tau)(\tau-t)}\mathrm{d}\tau \tag{38}$$

Finally we get the following theorem.

Theorem 3 The singular integral equations (20) with a parametric unknown function is surely solvable when the index $\kappa\geqslant 0$, and its general solution is $(\varphi(t),\psi(t))$, where $\varphi(t)$ and $\psi(t)$ are given by (35) and (36) respectively, in which $P_{\kappa-1}(z)$ is an arbitrary polynomial of degree $\kappa-1$ with $P_{-1}(z)\equiv 0$. When the index $\kappa<0$, it is solvable if and only if $-\kappa$ conditions given in (17) are satisfied. Under the above conditions, it has a unique solution $(\varphi(t),\psi(t))$, where $\varphi(t)$ and $\psi(t)$ are still given by (35) and (36) respectively, in which $P_{\kappa-1}(z)\equiv 0$. In a word, the general solution of the singular integral equations (20) with a parametric unknown function has κ degree of freedom.

Let $f_1(t)=f_2(t)=0$ in (20), we have the homogeneous singular integral equations with a parametric unknown function as follows

$$\begin{aligned}a_1(t)\varphi(t)+\frac{b_1(t)}{\pi i}\int_L\frac{\varphi(\tau)}{\tau-t}\mathrm{d}\tau=c_1(t)\psi(t)\\ a_2(t)\varphi(t)+\frac{b_2(t)}{\pi i}\int_L\frac{\varphi(\tau)}{\tau-t}\mathrm{d}\tau=c_2(t)\psi(t)\end{aligned}\qquad t\in L \tag{39}$$

According to the above results, the following theorem holds obviously.

Theorem 4 When the index $\kappa \geqslant 0$, the homogeneous singular integral equations (39) with a parametric unknown function possesses κ linearly independent solutions

$$(\varphi(t),\ \psi(t)) = \left(\frac{G_2(t) - G_1(t)}{G_2(t)} t^k X^+(t),\ \frac{D_1(t)G_1(t)}{G_2(t)} t^k X^+(t)\right) \qquad k = 0, 1, \cdots, \kappa - 1$$

When the index $\kappa < 0$, it only possesses zero-solution $(\varphi(t),\ \psi(t)) = (0,\ 0)$.

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奇异积分方程组与含参变未知函数的 Riemann 边值问题

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摘要: 提出多连通区域上含参变未知函数的 Riemann 边值问题, 给出其可解性定理和解的表示式, 然后使用上述结果, 给出了一类含参变未知函数的奇异积分方程组的新的解法, 获得了可解性定理和解的表示式.

关键词: 奇异积分方程组; Riemann 边值问题; 参变未知函数; 多连通区域

责任编辑 张 枸