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Existence of Homoclinic Orbits for a Class of Second-order Hamiltonian System with Coercive Potential①

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Abstract: The existence of homoclinic solution is obtained for second-order Hamiltonian systems $\ddot{u}(t)-\nabla V(t, u(t))=f(t)$, as the limit of a sequence of solutions for nil-boundary-value problems which are obtained via the least action principle.

Key words: critical point; coercive potential; second-order Hamiltonian system; homoclinic orbits

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Consider the second-order Hamiltonian systems

$$\ddot{u}(t)-\nabla V(t, u(t))=f(t) \tag{1}$$

where $V: R\times R^N \longrightarrow R$ and $f: R\longrightarrow R^N$ is a bounded function.

As well known, the homoclinic orbits are important in some applications. Via the critical point theory, the existence of homoclinic orbits and periodic solutions has been extensively studied by many authors (see References [1-7]). We say that a solution $u(t)$ of problem (1) is nontrivial homoclinic (to 0), if $u\neq 0$, $u(t)\rightarrow 0$ and $\dot{u}(t)\rightarrow 0$ as $t\rightarrow\pm\infty$. Here and subsequently, $(\cdot, \cdot): R^N\times R^N\longrightarrow R$ denotes the standard inner product in R^N, and $|\cdot|$ is the induced norm.

When $V(t, x)$ is periodic in t satisfying $V(t, x)\geqslant b|x|^2+V(t, 0)$, References [2] obtained a homoclinic solution of (1) as a limit in C^1_{loc}-topology of a certain sequence of $2kT$-periodic solutions. After then, References [6] obtained the same result with the periodicity of $V(t, x)$ in t and another weaker condition

$$V(t, x)\geqslant b|x|^{\mu}+V(t, 0)$$

where $\mu>1$. Motivated by these papers mentioned above, in this paper, without the periodicity of V, we obtain the homoclinic solutions of (1) which generalized the results in References [2, 6]. The main result is the following theorem.

Theorem 1　Suppose that $V: R\times R^N\longrightarrow R$ is a C^1-map, and for all $(t, x)\in R\times R^N$, the following conditions hold

(V_1) there are constants $a>0$, $\beta>1$, $\gamma\in(0, \beta)$ and a function $g\in L^{\frac{\beta}{\beta-\gamma}}(R, R^+)$ such that

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$$V(t,x) \geqslant a|x|^{\beta} - g(t)|x|^{\gamma} + V(t,0)$$

(V_2) $\nabla V(t,0)=0$ for all $t \in R$,

(V_3) for every $L>0$ the following inequality holds

$$\sup_{t\in R,\ |x|\leqslant L} |\nabla V(t,x)| < \infty$$

(f) $f \neq 0$ satisfies that $\int_R |f(t)|^{\frac{\beta}{\beta-1}} \mathrm{d}t < \infty$.

Then system (1) possesses at least one nontrivial homoclinic solution.

Remark 1 It is obviously that there exist functions V which satisfy our Theorem 1 without satisfying the corresponding assumptions in References [2, 6]. For example, let

$$V(t,x) = |x|^4 - \frac{1}{\sqrt{1+t^2}}|x|^2$$

We obtain the homoclinic solution of (1) as a limit in C^1_{loc}-topology of a certain sequence solutions. Fix $T>0$, in order to receive a homoclinic solution of (1), we consider a sequence of systems of differential equations

$$\begin{cases} \ddot{u}(t) - \nabla V(t,u(t)) = f(t) & t\in(-kT,kT) \\ u(-kT)=u(kT)=0 \end{cases} \tag{2}$$

We will prove that the existence of at least one homoclinic solution of (1) is the limit of the solutions of (2) as $k\to\infty$.

For each $k \in N$, set

$$E_k = \{u: [-kT, kT] \longrightarrow R^N \mid u \text{ is absolutely continuous}, u(-kT)=u(kT)=0\}$$

with the norm

$$\| u \|_{E_k} = \left(\int_{-kT}^{kT} (|\dot{u}(t)|^2 + |u(t)|^2)\mathrm{d}t\right)^{\frac{1}{2}}$$

and let $L^2_{2kT}(R, R^N)$ denote the Hilbert space of functions on R with values in R^N under the norm

$$\| u \|_{L^2_{2kT}(R,R^N)} = \left(\int_{-kT}^{kT} |u(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}}$$

Let $L^{\infty}_{2kT}(R, R^N)$ be a space of essentially bounded measurable functions from R into R^N under the norm

$$\| u \|_{L^{\infty}_{2kT}(R,R^N)} = \operatorname{esssup}\{|u(t)| : t\in[-kT,kT]\}$$

for every $k \in N$. Moreover, the corresponding function of (2) can be defined by

$$I_k(u) = \int_{-kT}^{kT} \left(\frac{1}{2}|\dot{u}(t)|^2 + V(t,u(t)) + (f_k(t), u(t))\right)\mathrm{d}t \tag{3}$$

Then one can easily check that $I_k \in C^1(E_k, R)$ is weakly lower semi-continuous and

$$\langle I_k'(u), v\rangle = \int_{-kT}^{kT} ((\dot{u}(t),\dot{v}(t)) + (\nabla V(t,u(t)), v(t)) + (f_k(t), v(t)))\mathrm{d}t \tag{4}$$

Furthermore, the critical points of I_k are classical solutions of (2). We divided the proof of Theorem 1 into a sequence of lemmas.

Lemma 1 Suppose that V and f satisfy (V_1) and (f), then system (2) possesses a solution $u_k \in E_k$ for every $k\in N$ such that

$$\left(\int_{-kT}^{kT} |\dot{u}_k(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} + \left(\int_{-kT}^{kT} |u_k(t)|^{\beta}\mathrm{d}t\right)^{\frac{1}{\beta}} \leqslant M_0 \tag{5}$$

for some $M_0>0$.

Proof Set $C_k = \int_{-kT}^{kT} V(t,0)\mathrm{d}t$. Since $\beta>\gamma$, by (V_1) and (f), we have

$$I_k(u) = \int_{-kT}^{kT}\left(\frac{1}{2}|\dot{u}(t)|^2 + V(t,u(t)) + (f_k(t),u(t))\right)\mathrm{d}t$$

$$
\begin{aligned}
&\geqslant \frac{1}{2}\int_{-kT}^{kT} |\dot{u}(t)|^2 \mathrm{d}t + a\int_{-kT}^{kT} |u(t)|^\beta \mathrm{d}t - \int_{-kT}^{kT} g(t)\,|u(t)|^\gamma \mathrm{d}t - \int_{-kT}^{kT} (f_k(t),\ u(t))\mathrm{d}t + C_k \\
&\geqslant \frac{1}{2}\int_{-kT}^{kT} |\dot{u}(t)|^2 \mathrm{d}t + \frac{a}{2}\int_{-kT}^{kT} |u(t)|^\beta \mathrm{d}t - \|g\|_{\frac{\beta}{\beta-\gamma}} \left(\int_{-kT}^{kT} |u(t)|^\beta \mathrm{d}t\right)^{\frac{\gamma}{\beta}} + \\
&\quad \left(\frac{a}{2}\int_{-kT}^{kT} |u(t)|^\beta \mathrm{d}t - C_1\left(\int_{-kT}^{kT} |u(t)|^\beta \mathrm{d}t\right)^{\frac{1}{\beta}}\right) + C_k \\
&\geqslant \frac{1}{2}\int_{-kT}^{kT} |\dot{u}(t)|^2 \mathrm{d}t + \frac{a}{2}\int_{-kT}^{kT} |u(t)|^\beta \mathrm{d}t - \|g\|_{\frac{\beta}{\beta-\gamma}} \left(\int_{-kT}^{kT} |u(t)|^\beta \mathrm{d}t\right)^{\frac{\gamma}{\beta}} - C_2 + C_k \\
&\geqslant \frac{1}{2}\int_{-kT}^{kT} |\dot{u}(t)|^2 \mathrm{d}t + \frac{a}{4}\int_{-kT}^{kT} |u(t)|^\beta \mathrm{d}t - C_3 + C_k \\
&\geqslant \min\left\{\frac{1}{2},\ \frac{a}{4}\right\}\left(\int_{-kT}^{kT} |\dot{u}(t)|^2 \mathrm{d}t + \int_{-kT}^{kT} |u(t)|^\beta \mathrm{d}t\right) - C_3 + C_k
\end{aligned}
\tag{6}
$$

for some $C_1, C_2, C_3 > 0$, which are independent of k. By Soblev's inequality and Hölder inequality, it is easy to verify that, for each $k \in N$, the following conditions are equivalent

(a) $\|u\|_{E_k} \to \infty$;

(b) $\int_{-kT}^{kT} |\dot{u}(t)|^2 \mathrm{d}t + |\bar{u}|^2 \to \infty$;

(c) $\int_{-kT}^{kT} |\dot{u}(t)|^2 \mathrm{d}t + \int_{-kT}^{kT} |u(t)|^p \mathrm{d}t \to \infty$.

for any $p > 1$, where

$$
\bar{u} = \frac{1}{2kT}\int_{-kT}^{kT} u(t)\mathrm{d}t \qquad \tilde{u} = u(t) - \bar{u}
$$

Then one has

$$
I_k(u) \to +\infty \qquad \|u\|_{E_k} \to \infty
$$

By Theorem 1.1 and Corollary 1.1 in References [10] we conclude that there exists $u_k \in E_k$ such that

$$
I_k(u_k) = \inf_{u \in E_k} I_k(u)
$$

for every $k \in N$. Furthermore, since

$$
I_k(0) = \int_{-kT}^{kT} V(t,\ 0)\mathrm{d}t = C_k
$$

we have $I_k(u_k) \leqslant C_k$. It follows from (6) that there is a constant $C_4 > 0$ such that

$$
\int_{-kT}^{kT} |\dot{u}_k(t)|^2 \mathrm{d}t + \int_{-kT}^{kT} |u_k(t)|^\beta \mathrm{d}t \leqslant C_4 \tag{7}
$$

then we have

$$
\left(\int_{-kT}^{kT} |\dot{u}_k(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}} \leqslant C_4^{\frac{1}{2}} \qquad \left(\int_{-kT}^{kT} |u_k(t)|^\beta \mathrm{d}t\right)^{\frac{1}{\beta}} \leqslant C_4^{\frac{1}{\beta}}
$$

which implies (5). The proof is completed.

Next we need to prove that $\{u_k\}$ is uniformly bounded, before this, we state an estimation made in References [6].

Lemma 2[6] Let $d > 0$, $p > 1$ and $u \in W^{1,2}(R,\ R^N)$. Then for every $t \in R$, the following inequality holds

$$
|u(t)| \leqslant \sqrt{\frac{d}{2}}\left(\int_{t-d}^{t+d} |\dot{u}(s)|^2 \mathrm{d}s\right)^{\frac{1}{2}} + (2d)^{-\frac{1}{p}}\left(\int_{t-d}^{t+d} |u(s)|^p \mathrm{d}s\right)^{\frac{1}{p}} \tag{8}
$$

Lemma 3 Let $u \in E_k$, then there exists a constant $M > 0$ such that

$$
\|u\|_{L_{2kT}^\infty} \leqslant M\left(\left(\int_{-(k+1)T}^{(k+1)T} |\dot{u}(s)|^2 \mathrm{d}s\right)^{\frac{1}{2}} + \left(\int_{-(k+1)T}^{(k+1)T} |u(s)|^\beta \mathrm{d}s\right)^{\frac{1}{\beta}}\right) \tag{9}
$$

Proof Set $d=T$, $p=\beta$ in (8), the conclusion can be obtianed inmmediatly.

Lemma 4 Let $u_k\in E_k$ be the solution of system (2) which satisfies (5). Then there is a constant $M_1>0$ independent of k such that

$$\|u_k\|_{L^\infty_{2kT}}\leqslant M_1 \tag{10}$$

for all $k\in N$.

Proof It is obviously to obtain this conclusion by combining (5) and (9).

Lemma 5 Let $u_k\in E_k$ be the solution of system (2) which satisfies (10) for all $k\in N$. Then there exists a subsequence $\{u_{k_j}\}$ of $\{u_k\}_{k\in N}$ convergenting to some u_0 in $C^1_{\rm loc}(R, R^N)$.

Proof In order to finish the proof via the Arzelà-Ascoli theorem, we divide our proof into two steps.

Firstly, we show that $\{\dot{u}_k\}_{k\in N}$ and $\{\ddot{u}_k\}_{k\in N}$ are uniformly bounded sequence. Since u_k is a solution of system (2), it follows that

$$\ddot{u}_k(t)=\nabla V(t, u_k(t))+f(t) \tag{11}$$

for every $t\in(-kT, kT)$, then we have

$$\begin{aligned}|\ddot{u}_k(t)|&\leqslant|\nabla V(t, u_k(t))|+|f(t)|\\&\leqslant|\nabla V(t, u_k(t))|+\sup_{t\in R}|f(t)|\end{aligned}$$

for $k\in N$. Since V is a C^1-map, it follows from (V_3) and (10) that there is a constant $M_2>0$ independent of k such that

$$\|\ddot{u}_k\|_{L^\infty_{2kT}}\leqslant M_2 \tag{12}$$

We suppose that $u_k(t)=(u_{k_1}(t), u_{k_2}(t), \cdots, u_{k_N}(t))$ for each $t\in R$. By the Mean Value theorem, there exits $t_{k_i}\in[t-1, t]$ for all $t\in R$, such that

$$\dot{u}_{k_i}(t_{k_i})=\int_{t-1}^{t}\dot{u}_{k_i}(s)\mathrm{d}s=u_{k_i}(t)-u_{k_i}(t-1)$$

for any $i\in\{1, 2, \cdots, N\}$. Then by (10) and (12) we have

$$\begin{aligned}|\dot{u}_{k_i}(t)|&=\left|\int_{k_i}^{t}\ddot{u}_{k_i}(s)\mathrm{d}s+\dot{u}_{k_i}(t_{k_i})\right|\\&\leqslant\int_{t-1}^{t}|\ddot{u}_{k_i}(s)|\mathrm{d}s+|\dot{u}_{k_i}(t_{k_i})|\\&\leqslant\int_{t-1}^{t}|\ddot{u}_k(s)|\mathrm{d}s+|u_{k_i}(t)-u_{k_i}(t-1)|\\&\leqslant M_2+2M_1\equiv M_3\end{aligned}$$

Consequently, there exists a constant $M_4>0$ such that

$$\|\dot{u}_k\|_{L^\infty_{2kT}}\leqslant M_4 \tag{13}$$

Secondly, we need to prove that $\{u_k\}_{k\in N}$ and $\{\dot{u}_k\}_{k\in N}$ are equicontinuous. Actually, by (12) we get

$$|\dot{u}_k(t_1)-\dot{u}_k(t_2)|\leqslant\left|\int_{t_2}^{t_1}\ddot{u}_k(s)\mathrm{d}s\right|\leqslant\int_{t_2}^{t_1}|\ddot{u}_k(s)|\mathrm{d}s\leqslant M_2|t_1-t_2|$$

for each $k\in N$ and $t_1, t_2\in R$, which shows that $\{\dot{u}_k\}_{k\in N}$ is equicontinuous, and $\{u_k\}_{k\in N}$ remains in the same way. Then there is a subsequence $\{u_{k_j}\}_{k\in N}$ convergent to u_0 in $C^1_{\rm loc}(R, R^N)$ by the Arzelà-Ascoli theorem.

Our next proof is to show that u_0 is the desired homoclinic solution of (1).

Lemma 6 Let $u_0: R\longrightarrow R^N$ be the same function in Lemma 5. Then u_0 is a homoclinic solution of problem (1).

Proof First, we will show that $u_0(t)$ satisfies (1). By Lemmas 1 and 5, we have $u_{k_j}\rightarrow u_0$ in $C^1_{\rm loc}(R, R^N)$ as $j\rightarrow\infty$, and

$$\ddot{u}_{k_j}(t)-\nabla V(t, u_{k_j}(t))=f(t)$$

for each $j \in N$ and $t \in (-k_j T, k_j T)$. Take $a, b \in R$ such that $a < b$. There exists $j_0 \in N$ such that

$$\ddot{u}_{k_j}(t) - \nabla V(t, u_{k_j}(t)) = f(t) \tag{14}$$

for all $j > j_0$ and $t \in [a, b]$. Since, for $j > j_0$, $\ddot{u}_{k_j}(t)$ is continuous in $[a, b]$ and $\ddot{u}_{k_j}(t) \to \nabla V(t, u_0(t)) + f(t)$ uniformly on $[a, b]$. So it follows that $\ddot{u}_{k_j}$ is a classical derivative of $\dot{u}_{k_j}$ in (a, b) for each $j > j_0$. Moreover, since $\dot{u}_{k_j} \to \dot{u}_0$ uniformly on $[a, b]$, we get

$$\ddot{u}_0(t) - \nabla V(t, u_0(t)) = f(t)$$

for all $t \in [a, b]$. Since a and b are arbitrary, we conclude that u_0 satisfies (1).

In the next step we will prove that $u_0(t) \to 0$ as $t \to \pm\infty$. By (7), for every $l \in N$, there exists $j_0 \in N$ such that

$$\int_{-lT}^{lT} |\dot{u}_{k_j}(t)|^2 \mathrm{d}t + \int_{-lT}^{lT} |u_{k_j}(t)|^\beta \mathrm{d}t \leqslant C_4$$

for any $j \geqslant j_0$. From this and Lemma 5 it follows that

$$\int_{-lT}^{lT} |\dot{u}_0(t)|^2 \mathrm{d}t + \int_{-lT}^{lT} |u_0(t)|^\beta \mathrm{d}t \leqslant C_4$$

Letting $l \to \infty$, we obtain

$$\int_{-\infty}^{\infty} (|\dot{u}_{k_j}(t)|^2 + |u_{k_j}(t)|^\beta) \mathrm{d}t \leqslant C_4 \tag{15}$$

Hence

$$\int_{|t| \geqslant r} (|\dot{u}_0(t)|^2 + |u_0(t)|^\beta) \mathrm{d}t \to 0 \tag{16}$$

as $r \to +\infty$, which implies

$$\left(\int_{|t| \geqslant r} |\dot{u}_0(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}} \to 0 \qquad \left(\int_{|t| \geqslant r} |u_0(t)|^\beta \mathrm{d}t\right)^{\frac{1}{\beta}} \to 0 \tag{17}$$

By (8) we obtain

$$|u_0(t)| \leqslant \frac{\sqrt{2}}{2}\left(\int_{t-1}^{t+1} |\dot{u}_0(s)|^2 \mathrm{d}s\right)^{\frac{1}{2}} + 2^{-\frac{1}{\beta}}\left(\int_{t-1}^{t+1} |u_0(s)|^\beta \mathrm{d}s\right)^{\frac{1}{\beta}} \tag{18}$$

Combining (17) and (18), one has $u_0(t) \to 0$ as $t \to \pm\infty$.

Finally, we will show that $\dot{u}_0(t) \to 0$ as $t \to \pm\infty$. From Lemma 4 and Lemma 5, we obtain

$$|u_0(t)| \leqslant M_1 \tag{19}$$

for all $t \in R$. Since f is bounded and by (V_3), we have

$$\begin{aligned} |\ddot{u}_0(t)| &\leqslant |\nabla V(t, u_0(t))| + |f(t)| \\ &\leqslant \sup_{(t,x) \in [-kT, kT] \times [-M_1, -M_1]} |\nabla V(t, x)| + \sup_{t \in R} |f(t)| \\ &\equiv M_5 \end{aligned}$$

for all $t \in R$. If our claim is false, there exist $\varepsilon_0 > 0$ and a sequence $\{t_k\}$ such that

$$|t_1| < |t_2| < |t_3| < \cdots \qquad |t_k| + 1 < |t_{k+1}| \qquad k = 1, 2, \cdots$$

and

$$|\dot{u}_0(t_k)| \geqslant 2\varepsilon_0 \qquad k = 1, 2, \cdots$$

From this we obtain

$$|\dot{u}_0(t)| = \left|\int_{t_k}^{t} \ddot{u}_0(s) \mathrm{d}s + \dot{u}_0(t_k)\right| \geqslant |\dot{u}_0(t_k)| - \int_{t_k}^{t} |\ddot{u}_0(s)| \mathrm{d}s \geqslant \varepsilon_0$$

for all $t \in [t_k, t_k + \varepsilon_0(1 + M_5)]$, which implies

$$\int_{-\infty}^{\infty} |\dot{u}_0(t)|^2 \mathrm{d}t \geqslant \sum_{k=1}^{\infty} \int_{t_k}^{t_k + \frac{\varepsilon_0}{(1+M_5)}} |\dot{u}_0(t)|^2 \mathrm{d}t = \infty$$

which contradicts (15) and then we obtain our conclusion. Since $\nabla V(t, 0) = 0$ for all in $t \in R$, which together with (f), we obtain $u=0$ is not the solution of (1), hence $u_0 \neq 0$. The proof is completed.

Proof of Theorem 1

By Lemma 1 to Lemma 6, Theorem 1 is proved.

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一类带强制位势的二阶 Hamilton 系统的同宿轨的存在性

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摘要： 通过对一列由最小作用原理得到的零边值问题的解取极限，得到了二阶哈密尔顿系统 $\ddot{u}(t)-\nabla V(t, u(t))=f(t)$ 同宿轨的存在性结论.

关键词： 临界点；强制位势；二阶 Hamilton 系统；同宿轨

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