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A New General Iterative Method for Continuous Pseudocontractive Mappings in Hilbert Spaces①

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Abstract: The purpose of this paper is to prove that the sequence $\{x_n\}$ generated by the iterative method

$$x_n = \alpha_n \gamma f(x_{n-1}) + (I - \alpha_n A) T x_n \qquad n \geqslant 0$$

converges strongly to a fixed point $p \in F(T)$ which solves the variational inequality

$$\langle (\gamma f - A) p, y - p \rangle \leqslant 0 \qquad y \in F(T)$$

Key words: continuous pseudocontractive mapping; strongly positive linear bounded operator; variational inequality

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Let C be a nonempty closed convex subset of a real Hilbert space H and T be a continuous pseudocontractive mapping from C into itself, and A be a strongly positive linear bounded operator on H. Denote by $F(T)$ the set of fixed points of T and assume that $F(T)$ be nonempty. In 2006, reference [1] introduced an iterative scheme of continuous pseudocontractive mapping and proved the iterative scheme converge strongly to the unique solution $p \in C$ of the variational inequality. The aim of this paper is to introduce a new general iterative method and to prove the general iterative method converge strongly to the unique solution $p \in C$ of the variational inequality. This results extend some corresponding results.

Now let T be a pseudocontractive mapping, f be a strongly pseudocontractive mapping with coefficient $\alpha \in (0, 1)$ and $t \in (0, 1)$ such that $t < \| A \|^{-1}$ and $0 < \gamma < \frac{\beta}{\alpha}$. Consider a mapping W_t on H defined by

$$W_t x = t\gamma f(x) + (I - tA) T x \qquad x \in H$$

It is easy to proved that W_t is a strongly pseudocontractive mapping by Lemma 2.2 in reference [1], and W_t has a unique fixed point, denoted x_t, which uniquely solves the fixed point equation

$$x_t = t\gamma f(x_t) + (I - tA) T x_t$$

from Corollary 2 in reference [2]. Therefore, x_t is well defined.

Theorem 1　Let H be a Hilbert space, and K be a nonempty compact convex subset of H. Suppose that $T: K \longrightarrow K$ is a continuous pseudocontractive mapping with $F(T) \neq \varnothing$, $A: K \longrightarrow K$ is a strongly positive linear bounded operator with coefficient $\beta > 0$, and $f: K \longrightarrow K$ is a fixed Lipschitzian strongly

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pseudocontractive mapping with pseudocontractive coefficient $\alpha \in (0, 1)$ and Lipschitzian constant $L > 0$. Let x_t be defined by

$$x_t = t\gamma f(x_t) + (I - tA)Tx_t \tag{1}$$

If $0 < \gamma < \frac{\beta}{\alpha}$, then as $t \to 0$, x_t converges strongly to some fixed point p of T which solves the variational inequality:

$$\langle (A - \gamma f)p, p - y\rangle \leqslant 0 \qquad y \in F(T) \tag{2}$$

Proof First, we show the uniqueness of a solution to the variational inequality (2). In fact, suppose that $p, q \in F(T)$ satisfy inequality (2), we get that

$$\langle (A - \gamma f)p, p - q\rangle \leqslant 0 \tag{3}$$

$$\langle (A - \gamma f)q, q - p\rangle \leqslant 0 \tag{4}$$

Adding up inequalities (3) and (4), we have

$$\langle (A - \gamma f)p - (A - \gamma f)q, p - q\rangle \leqslant 0$$

From Lemma 2.2 in reference [1], the strong monotonicity of $A - \gamma f$ implies that $p = q$, and the uniqueness is proved. We use $p \in F(T)$ to denote the unique solution of inequality (2).

Since $t \to 0$, we may assume, with no loss of generality, that $t < \| A \|^{-1}$. We have $\| I - tA \| \leqslant 1 - t\beta$ by Lemma 2.3 in reference [1]. Take a fixed point $p \in F(T)$, we have

$$\| x_t - p \|^2 \leqslant (1 - t(\beta - \gamma\alpha)) \| x_t - p \|^2 + t\langle \gamma f(p) - Ap, x_t - p\rangle$$

Hence

$$t(\beta - \gamma\alpha) \| x_t - p \|^2 \leqslant t \| \gamma f(p) - Ap \| \| x_t - p \|$$

$$\| x_t - p \| \leqslant \frac{1}{\beta - \gamma\alpha} \| \gamma f(p) - Ap \|$$

So $\{x_t : 0 < t < \| A \|^{-1}\}$ is bounded.

As f is a Lipschitzian mapping, $\| f(x_t) - f(p) \| \leqslant L \| x_t - p \|$, then the set $\{f(x_t) : t \in (0, \| A \|^{-1})\}$ is bounded. By $x_t = t\gamma f(x_t) + (I - tA)Tx_t$, we have that

$$-(I - tA)Tx_t = t\gamma f(x_t) - x_t \qquad \| (I - tA)Tx_t \| \leqslant t\gamma \| f(x_t) \| + \| x_t \|$$

Therefore, the set $\{(I - tA)Tx_t\}$ is also bounded by the boundedness of $\{x_t\}$ and $\{f(x_t)\}$. Since the operator A is bounded, then $\{ATx_t\}$ is also bounded, which implies that

$$\lim_{t \to 0} \| x_t - Tx_t \| = \lim_{t \to 0} t \| \gamma f(x_t) - ATx_t \| = 0 \tag{5}$$

Since $\{x_t\}$ is bounded and K is a nonempty compact convex subset of H, then there exists a subsequence, denoted by $\{x_{t_n}\} \subset \{x_t\}$, where $t_n \to 0$ as $n \to \infty$, such that $x_{t_n} \to q$ as $n \to \infty$. Moreover, by formula (5), we have $q \in F(T)$. We next show that q solves the variational inequality

$$\langle (A - \gamma f)p, p - y\rangle \leqslant 0 \qquad y \in F(T)$$

From formula (1), we have

$$-tAx_t + t\gamma f(x_t) = (I - tA)x_t - (I - tA)Tx_t$$

hence

$$(A - \gamma f)x_t = -\frac{1}{t}(I - tA)(I - T)x_t$$

It follows that, for any $y \in F(T)$, one has

$$\langle (A - \gamma f)x_t, x_t - y\rangle = -\frac{1}{t}\langle (I - tA)(I - T)x_t, x_t - y\rangle \leqslant \langle A(I - T)x_t, x_t - y\rangle \tag{6}$$

Due to the continuous pseudocontractive mapping T, $I - T$ is monotone, i.e., $\langle x - y, (I - T)x - (I - T)y\rangle \geqslant 0$ for $x, y \in H$. Now replace t in inequality (6) with t_n, and let $n \to \infty$, notice that

$$\lim_{n\to\infty}(I-T)x_{t_n}=(I-T)q=0 \qquad q\in F(T)$$

we obtain

$$\langle (A-\gamma f)q,\ q-y\rangle \leqslant 0$$

That is, $q\in F(T)$ is a solution of inequality (2), hence $q=p$. So we have that each cluster point of $\{x_t\}$ (as $t\to 0$) equals p. Therefore, $x_t\to p$ as $t\to 0$. The proof is complete.

Theorem 2 Let H be a Hilbert space, and K a nonempty compact convex subset of H. Suppose that $T: K\longrightarrow K$ is a continuous pseudocontractive mapping with $F(T)\neq\varnothing$, $A: K\longrightarrow K$ is a strongly positive linear bounded operator with coefficient $\beta>0$ and $f: K\longrightarrow K$ is a fixed contractive mapping with contractive coefficient $\alpha\in(0,1)$. For $x_0\in K$, $\{x_n\}$ is given by

$$x_n=\alpha_n\gamma f(x_{n-1})+(I-\alpha_n A)Tx_n \tag{7}$$

If $\{\alpha_n\}\in(0,1)$ such that $\alpha_n\to 0$ $(n\to\infty)$ and $0<\gamma<\dfrac{\beta}{\alpha}$, then $\{x_n\}$ converges strongly to a fixed point p of T such that p is unique solution in $F(T)$ of the variational inequality (2).

Proof First, we observe that $\{x_n\}$ is bounded. Indeed, take a fixed $p\in F(T)$, we have

$$\begin{aligned}
&\|x_n-p\|^2=\\
&\langle (I-\alpha_n A)(Tx_n-p)+\alpha_n(\gamma f(x_{n-1})-Ap),\ x_n-p\rangle \leqslant\\
&(1-\alpha_n\beta)\|x_n-p\|^2+\alpha_n\langle \gamma f(x_{n-1})-Ap,\ x_n-p\rangle =\\
&(1-\alpha_n\beta)\|x_n-p\|^2+\alpha_n\langle \gamma f(x_{n-1})-\gamma f(p)+\gamma f(p)-Ap,\ x_n-p\rangle =\\
&(1-\alpha_n\beta)\|x_n-p\|^2+\alpha_n(\langle \gamma f(x_{n-1})-\gamma f(p),\ x_n-p\rangle+\langle \gamma f(p)-Ap,\ x_n-p\rangle) \leqslant\\
&(1-\alpha_n\beta)\|x_n-p\|^2+\alpha_n\gamma\alpha\|x_{n-1}-p\|\ \|x_n-p\|+\alpha_n\langle \gamma f(p)-Ap,\ x_n-p\rangle
\end{aligned} \tag{8}$$

It follows from inequalities (8) that

$$\|x_n-p\|\leqslant\frac{\gamma\alpha}{\beta}\|x_{n-1}-p\|+\frac{1}{\beta}\|\gamma f(p)-Ap\|=\frac{\gamma\alpha}{\beta}\|x_{n-1}-p\|+\frac{\beta-\gamma\alpha}{\beta}\frac{\|\gamma f(p)-Ap\|}{\beta-\gamma\alpha}$$

By induction, we have

$$\|x_n-p\|\leqslant\max\left\{\|x_0-p\|,\ \frac{\|\gamma f(p)-Ap\|}{\beta-\gamma\alpha}\right\}$$

$$\|f(x_n)\|\leqslant\|f(x_n)-f(p)\|+\|f(p)\|\leqslant\alpha\|x_n-p\|+\|f(p)\|$$

which imply that $\{x_n\}$ and $\{f(x_n)\}$ are bounded. By $x_n=\alpha_n\gamma f(x_{n-1})+(I-\alpha_n A)Tx_n$, we have

$$-(I-\alpha_n A)Tx_n=\alpha_n\gamma f(x_{n-1})-x_n \qquad \|(I-\alpha_n A)Tx_n\|\leqslant\alpha_n\gamma\|f(x_{n-1})\|+\|x_n\|$$

Therefore, the set $\{(I-\alpha_n A)Tx_n\}$ is also bounded by the boundedness of $\{x_n\}$ and $\{f(x_{n-1})\}$. Since the operator A is bounded, then $\{ATx_n\}$ is also bounded. From $\alpha_n\to 0$ $(n\to\infty)$, we have

$$\|x_n-Tx_n\|=\alpha_n\|\gamma f(x_{n-1})-ATx_n\|\to 0 \qquad n\to\infty \tag{9}$$

Then from inequalities (8), we have

$$\begin{aligned}
\beta\|x_n-p\|^2&\leqslant\gamma\alpha\|x_{n-1}-p\|\ \|x_n-p\|+\langle \gamma f(p)-Ap,\ x_n-p\rangle\leqslant\\
&\frac{\gamma\alpha}{2}(\|x_{n-1}-p\|^2+\|x_n-p\|^2)+\langle \gamma f(p)-Ap,\ x_n-p\rangle
\end{aligned}$$

Therefore

$$\|x_n-p\|^2\leqslant\left(1-\frac{2\beta-2\gamma\alpha}{2\beta-\gamma\alpha}\right)\|x_{n-1}-p\|^2+\frac{2\beta-2\gamma\alpha}{2\beta-\gamma\alpha}\frac{1}{\beta-\gamma\alpha}\langle \gamma f(p)-Ap,\ x_n-p\rangle$$

That is

$$\|x_n-p\|^2\leqslant(1-\gamma_n)\|x_{n-1}-p\|^2+\gamma_n\varepsilon_n$$

where $\gamma_n=\dfrac{2\beta-2\gamma\alpha}{2\beta-\gamma\alpha}$, $\varepsilon_n=\dfrac{1}{\beta-\gamma\alpha}\langle \gamma f(p)-Ap,\ x_n-p\rangle$. Clearly $\gamma_n\in(0,1)$ and $\sum\limits_{n=1}^{\infty}\gamma_n=\infty$. So we only need to show that $\limsup_{n\to\infty}\varepsilon_n\leqslant 0$, that is

$$\limsup_{n\to\infty}\langle\gamma f(p)-Ap,\ x_n-p\rangle\leqslant 0 \tag{10}$$

To see this, we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle\gamma f(p)-Ap,\ x_n-p\rangle=\lim_{k\to\infty}\langle\gamma f(p)-Ap,\ x_{n_k}-p\rangle$$

Since K is compact convex subset of H and $\{x_n\}$ is bounded, then there exists a subsequence $\{x_{n_k}\}\subset\{x_n\}$ such that $x_{n_k}\to y\in K$. By the continuity of the mapping T and the norm $\|\cdot\|$, together with formula (9), we have

$$\|y-Ty\|=\lim_{k\to 0}\|x_{n_k}-Tx_{n_k}\|=0$$

Therefore $y\in F(T)$. It follows from the variational inequality (2) that

$$\limsup_{n\to\infty}\langle\gamma f(p)-Ap,\ x_n-p\rangle=\langle\gamma f(p)-Ap,\ y-p\rangle\leqslant 0$$

So inequality (10) hold. By Lemma 2.4 in reference [3], $x_n\to p$.

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希尔伯特空间中一类新的连续伪压缩映射的广义迭代算法

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摘要:在希尔伯特空间中研究了一类新的连续伪压缩映射的广义迭代过程

$$x_n=\alpha_n\gamma f(x_{n-1})+(I-\alpha_n A)Tx_n \qquad n\geqslant 0$$

并证明了由该迭代算法生成的序列$\{x_n\}$的收敛点为变分不等式

$$\langle(\gamma f-A)p,\ y-p\rangle\leqslant 0 \qquad y\in F(T)$$

的解.

关键词: 连续伪压缩映射; 强正线性有界算子; 变分不等式

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