第 37 卷　第 4 期　　西南师范大学学报(自然科学版)　　2012 年 4 月
Vol. 37　No. 4　Journal of Southwest China Normal University (Natural Science Edition)　Apr.　2012

文章编号:1000-5471(2012)04-0020-04

Some Characterizations of Maximal Prefix Sets①

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Abstract: Let $M(X)$ be the monoid of language over an alphabet X. Some characterizations of maximal prefix sets of $M(X)$ are given.

Key words: prefix set; maximal prefix set; left cancellative subsemigroup

CLC number: O152.7　　**Document code**: A

Let X be an alphabet and X^* be the free monoid generated by X. Let $X^+=X^*\setminus\{1\}$, where 1 is the empty word. Every element of X^* is called a word. Every subset of X^* is called a language. The length of a word x is denoted by $lg(x)$ and the length of a language A is denoted by $lg(A)=\max\{lg(x)\mid x\in A\}$. The concatenation of two languages A and B is the set $AB=\{xy\mid x\in A, y\in B\}$. The family of languages $M(X)=\{A\mid A\subseteq X^+ \text{ or } A=\{1\}\}$ with the concatenation operation is a monoid with the identity $\{1\}$ (see reference [1]). We call $M(X)$ the monoid of language over X.

A nonempty subset α of $M(X)$ is called a prefix set if $A, AB\in\alpha$ imply $B=\{1\}$. A prefix set α is called a maximal prefix set (or prefix exhaustive set) if any prefix set $\beta\subseteq M(X)$ and $\alpha\subseteq\beta$ imply $\alpha=\beta$. Note that α is a maximal prefix set if and only if for any $H\in M(X)\setminus\alpha$, either $H=AU$ for some $A\in\alpha$, $U\in M(X)\setminus\{\{1\}\}$ or $HV\in\alpha$ for some $V\in M(X)$ (see reference [1]). In reference [2], some properties of prefix sets in the monoid of languages $M(X)$ were studied. The purpose of this paper is to give some characterizations of maximal prefix set. As in reference [1], we use the following notations:

For any $\alpha,\beta\subseteq M(X)$, we define

$$\alpha\beta=\{AB\mid A\in\alpha, B\in\beta\}$$

$$\alpha^*=\bigcup_{i=0}^{\infty}\alpha^i\qquad \alpha^+=\bigcup_{i=1}^{\infty}\alpha^i\qquad \alpha^0=\{\{1\}\}$$

$$T_\alpha=\{A\in M(X)\mid \text{exists } B\in M(X) \text{ such that } AB\in\alpha\}\qquad T_\alpha^C=M(X)\setminus T_\alpha$$

For standard terms and concepts in code theory, one may consult reference [2] or [3].

Remark 1　For convenience, let $\bar{1}$ denote $\{1\}$. Let α be a (maximal) prefix set. If $\bar{1}\in\alpha$, then by the definition of (maximal) prefix set, $\alpha=\{\bar{1}\}$. Throughout this paper, we always assume that $\bar{1}\notin\alpha$, i.e., $\alpha\neq\{\bar{1}\}$ when α is a (maximal) prefix set. If $\bar{1}\notin\alpha$, it is obvious that α is a prefix set if and only if $\alpha(M(X)\setminus\{\bar{1}\})\cap\alpha=\varnothing$.

Theorem 1　Let α be a prefix set. Then α is a maximal prefix set if and only if $T_\alpha^C\subseteq\alpha(M(X)\setminus\{\bar{1}\})$.

Proof　Let α be a maximal prefix set. We note that α is a maximal prefix set if and only if for any $A\in$

① 收稿日期:2010-11-18
基金项目:贵州省科技基金(黔科合[2010]3174).
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$M(X)\backslash\alpha$, either $A=A_1U$ for some $A_1\in\alpha$, $U\in M(X)\backslash\{\bar{1}\}$ or $AV\in\alpha$ for some $V\in M(X)$. Let $A\in T_\alpha^C$, then by the definition of T_α^C, $A=A_1U$ for some $A_1\in\alpha$, $U\in M(X)\backslash\{\bar{1}\}$. Thus $T_\alpha^C\subseteq\alpha(M(X)\backslash\{\bar{1}\})$.

Conversely, assume that $T_\alpha^C\subseteq\alpha(M(X)\backslash\{\bar{1}\})$. If α is not a prefix set, then there exists $\beta\subseteq M(X)$ such that $\alpha\subset\beta$, and β is a prefix set. Let $A\in\beta\backslash\alpha$. If $A\in T_\alpha$, then there exists $B\in M(X)$ such that $AB\in\alpha\subset\beta$. Since β is a prefix set, we have $B=\{\bar{1}\}$. Then $A=AB\in\alpha$, a contradiction. If $A\in T_\alpha^C$, since $T_\alpha^C\subseteq\alpha(M(X)\backslash\{\bar{1}\})$, then there exist $A_1\in\alpha$ and $U\in M\backslash\{\bar{1}\}$ such that $A=A_1U$. Note that $A_1\in\alpha\subset\beta$ and $A_1U=A\in\beta$. Since β is a prefix set, we have $U=\{\bar{1}\}$, a contradiction.

Lemma 1 Let $\alpha\subseteq M(X)$. Then α is a prefix set if and only if $T_\alpha\cap\alpha(M(X)\backslash\{\bar{1}\})=\emptyset$ (i. e., $\alpha(M(X)\backslash\{\bar{1}\})\subseteq T_\alpha^C$).

Proof Let α be a prefix set. If $T_\alpha\cap\alpha(M(X)\backslash\{\bar{1}\})\neq\emptyset$, then there exist $A\in T_\alpha$, $A_1\in\alpha$ and $U\in(M(X)\backslash\{\bar{1}\})$ such that $A=A_1U$. Since $A\in T_\alpha$, then there exists $B\in M(X)$ such that $AB\in\alpha$. Note that $A_1\in\alpha$ and $A_1UB=AB\in\alpha$. Since α is a prefix set, we have $UB=\{\bar{1}\}$. Thus $U=\{\bar{1}\}$, a contradiction.

Conversely, assume that $T_\alpha\cap\alpha(M(X)\backslash\{\bar{1}\})=\emptyset$. If α is not a prefix set, then there exist $A\in\alpha$ and $B\in M(X)\backslash\{\bar{1}\}$ such that $AB\in\alpha$. Since $\alpha\subseteq T_\alpha$, we have $AB\in T_\alpha\cap\alpha(M(X)\backslash\{\bar{1}\})$, i. e., $T_\alpha\cap\alpha(M(X)\backslash\{\bar{1}\})\neq\emptyset$, a contradiction. Thus α is a prefix set.

From Theorem 1 and Lemma 1, we easily deduce:

Theorem 2 Let $\alpha\subseteq M(X)$. Then α is a maximal prefix set if and only if $T_\alpha^C=\alpha(M(X)\backslash\{\bar{1}\})$.

The following Theorem is very useful in the rest of this paper.

Theorem 3 Let α be a prefix set. Then α is a maximal prefix set if and only if $M(X)=\alpha^* T_\alpha$.

Proof Let α be a maximal prefix set. To prove $M(X)=\alpha^* T_\alpha$, it is sufficient to show that $M(X)\subseteq\alpha^* T_\alpha$. Clearly, $\bar{1}\in\alpha^* T_\alpha$. Let $A\in M(X)\backslash\{\bar{1}\}$. If $A\in T_\alpha$, then $A\in T_\alpha\subseteq\alpha^* T_\alpha$. If $A\in T_\alpha^C$, then $A\in\alpha(M(X)\backslash\{\bar{1}\})$ by Theorem 2. Thus, there exist $B\in\alpha$ and $V_1\in M(X)\backslash\{\bar{1}\}$ such that $A=BV_1$ with $lg(V_1)<lg(A)$ (since $\bar{1}\notin\alpha$ by Remark 1). If $V_1\in T_\alpha$, then $A=BV_1\in\alpha T_\alpha\subseteq\alpha^* T_\alpha$. If $V_1\in T_\alpha^C$, then $V_1\in\alpha(M(X)\backslash\{\bar{1}\})$ by Theorem 2. Thus, there exist $B_1\in\alpha$ and $V_2\in M(X)\backslash\{\bar{1}\}$ such that $V_1=B_1V_2$ with $lg(V_2)<lg(V_1)$. Then $A=BB_1V_2$. Continuing this demonstration, we will have

$$A=BB_1\cdots B_{n-1}V_n \qquad B_i\in\alpha,\ V_n\in T_\alpha,\ n\geqslant 2$$

Otherwise, we will have

$$A=BB_1\cdots B_{n-1}V_m \qquad B_i\in\alpha,\ V_m\in T_\alpha^C,\ lg(V_m)=1,\ m\geqslant 2$$

Note that $\bar{1}\notin\alpha$ by Remark 1. By Theorem 2, we have $V_m\in\alpha(M(X)\backslash\{\bar{1}\})$, and so $lg(V_m)\geqslant 2$, a contradiction. Then $A=BB_1\cdots B_{n-1}V_n\subseteq\alpha^+ T_\alpha\subseteq\alpha^* T_\alpha$. Thus $M(X)=\alpha^* T_\alpha$.

Conversely, assume that $M(X)=\alpha^* T_\alpha$. Since $T_\alpha^C\subseteq M(X)=\alpha^* T_\alpha$, we have

$$T_\alpha^C\subseteq\alpha^+ T_\alpha\subseteq\alpha\,\alpha^* T_\alpha\subseteq\alpha M(X)$$

By the definition of T_α^C, we have $T_\alpha^C\cap\alpha=\emptyset$. It follows that

$$T_\alpha^C\subseteq\alpha M(X)\backslash\alpha\subseteq\alpha(M(X)\backslash\{\bar{1}\})$$

Thus, by Theorem 1, α is a maximal prefix set.

Lemma 2 Let α be a prefix set. Then $T_\alpha\cap(\alpha^+ T_\alpha\backslash\alpha)=\emptyset$.

Proof By Lemma 1, we have $T_\alpha\cap\alpha(M(X)\backslash\{\bar{1}\})=\emptyset$. Suppose that $T_\alpha\cap(\alpha^+ T_\alpha\backslash\alpha)\neq\emptyset$. Then there exist $B\in T_\alpha\backslash\alpha$, $C\in T_\alpha$, $A_1, A_2, \cdots, A_r\in\alpha$, $r\geqslant 1$, such that $B=A_1A_2\cdots A_rC$. If $r\geqslant 2$, then

$$B=A_1(A_2\cdots A_rC)\in T_\alpha\cap\alpha(M(X)\backslash\{\bar{1}\})$$

If $r=1$ and $B=A_1C$, since $B\notin\alpha$, we have $C\neq\bar{1}$. Then $B=A_1C\in T_\alpha\cap\alpha(M(X)\backslash\{\bar{1}\})$, thus $T_\alpha\cap\alpha(M(X)\backslash$

$\{\bar{1}\})\neq\varnothing$, a contradiction.

By Theorem 3, we can easily obtain the following Theorem:

Theorem 4 Let α be a prefix set. Then α is a maximal prefix set if and only if $T_\alpha^C=\alpha^+T_\alpha\backslash\alpha$.

Proof Note that $\alpha\subseteq T_\alpha$. Let α be a maximal prefix set. Then, by Theorem 3,

$$M(X)=\alpha^*T_\alpha=T_\alpha\cup\alpha^+T_\alpha=T_\alpha\cup(\alpha^+T_\alpha\backslash\alpha)$$

Thus, by Lemma 2, $T_\alpha^C=\alpha^+T_\alpha\backslash\alpha$. Conversely, assume that $T_\alpha^C=\alpha^+T_\alpha\backslash\alpha$. Then

$$M(X)=T_\alpha\cup T_\alpha^C=T_\alpha\cup(\alpha^+T_\alpha\backslash\alpha)=T_\alpha\cup\alpha^+T_\alpha=\alpha^*T_\alpha$$

Thus, by Theorem 3, α is a maximal prefix set.

Remark 2 In general, Theorem 4 is false without the assumption that α is a prefix set. Let $X=\{a,b\}$ and $\alpha=M(X)\backslash\{\bar{1}\}$. Then $T_\alpha=M(X)$ and $\alpha=\alpha^+T_\alpha$. Thus $T_\alpha^C=\alpha^+T_\alpha\backslash\alpha=\varnothing$. Since $\{a^2\}=\{a\}\{a\}\in\alpha\cap\alpha(M(X)\backslash\{\bar{1}\})$, then α is not a prefix set, and so α is not a maximal prefix set.

However, we have:

Theorem 5 Let $\alpha\subseteq M(X)$, then α is a maximal prefix set if and only if $T_\alpha^C=\alpha^+(T_\alpha\backslash\{\bar{1}\})$.

To prove Theorem 5 we need the following Lemmas:

Lemma 3 Let $\alpha\subseteq M(X)\backslash\{\bar{1}\}$. Then $\alpha^+T_\alpha\backslash\alpha\subseteq\alpha^+(T_\alpha\backslash\{\bar{1}\})$.

Proof Let $A\in\alpha^+T_\alpha\backslash\alpha$, then there exist $A_1\in\alpha^+$ and $U\in T_\alpha$ such that $A=A_1U$. If $U\neq\bar{1}$, then $A=A_1U\in\alpha^+(T_\alpha\backslash\{\bar{1}\})$. If $U=\bar{1}$, since $A\notin\alpha$, we have $A=A_1\in\alpha^+\backslash\alpha$. Note that

$$\alpha^+\backslash\alpha=\bigcup_{i=1}^{\infty}\alpha^i\backslash\alpha\subseteq\bigcup_{i=2}^{\infty}\alpha^i=(\bigcup_{i=1}^{\infty}\alpha^i)\alpha=\alpha^+\alpha$$

By the definition of T_α and $\bar{1}\notin\alpha$, we have $\alpha\subseteq T_\alpha\backslash\{\bar{1}\}$. Then $A\in\alpha^+\backslash\alpha\subseteq\alpha^+\alpha\subseteq\alpha^+(T_\alpha\backslash\{\bar{1}\})$. Thus $\alpha^+T_\alpha\backslash\alpha\subseteq\alpha^+(T_\alpha\backslash\{\bar{1}\})$.

Remark 3 In general, $\alpha^+(T_\alpha\backslash\{\bar{1}\})\subseteq\alpha^+T_\alpha\backslash\alpha$ is false. Let $X=\{a,b\}$ and $\alpha=\{\{a\},\{a^2\}\}$, then $T_\alpha=\alpha\cup\{\bar{1}\}$ and $\{a^2\}=\{a\}\{a\}\in\alpha\alpha\subseteq\alpha^+\alpha=\alpha^+(T_\alpha\backslash\{\bar{1}\})$. Since $\{a^2\}\in\alpha$, we have $\{a^2\}\notin\alpha^+T_\alpha\backslash\alpha$.

Lemma 4 Let α be a prefix set, then $\alpha^+T_\alpha\backslash\alpha=\alpha^+(T_\alpha\backslash\{\bar{1}\})$.

Proof By Lemma 3, it is sufficient to show that $\alpha^+(T_\alpha\backslash\{\bar{1}\})\subseteq\alpha^+T_\alpha\backslash\alpha$. Since $\bar{1}\notin\alpha$ (by Remark 1), we have $\alpha^*(T_\alpha\backslash\{\bar{1}\})\subseteq M(X)\backslash\{\bar{1}\}$. Then

$$\alpha^+(T_\alpha\backslash\{\bar{1}\})\cap\alpha=\alpha(\alpha^*(T_\alpha\backslash\{\bar{1}\}))\cap\alpha\subseteq\alpha(M(X)\backslash\{\bar{1}\})\cap\alpha=\varnothing$$

since α is a prefix set. Then $\alpha^+(T_\alpha\backslash\{\bar{1}\})=\alpha^+(T_\alpha\backslash\{\bar{1}\})\backslash\alpha\subseteq\alpha^+T_\alpha\backslash\alpha$.

Remark 4 In general, the converse of Lemma 4 is false. Let $X=\{a,b\}$ and $\alpha=\{\{a\},\{ab\}\}$, then $T_\alpha=\alpha\cup\{\bar{1}\}$ and $\alpha=T_\alpha\backslash\{\bar{1}\}$. It is obvious that $\alpha^+\alpha\cap\alpha=\varnothing$ and $\alpha^+=\alpha\cup\alpha^+\alpha$, we have $\alpha^+\backslash\alpha=\alpha^+\alpha$. Thus

$$\alpha^+T_\alpha\backslash\alpha=\alpha^+(\alpha\cup\{\bar{1}\})\backslash\alpha=\alpha^+\backslash\alpha=\alpha^+\alpha=\alpha^+(T_\alpha\backslash\{\bar{1}\})$$

Since $\{ab\}=\{a\}\{b\}\in\alpha\cap\alpha(M\backslash\{\bar{1}\})$, we have that α is not a prefix set.

Proof of Theorem 5 Let α be a maximal prefix set. By Theorem 4 and Lemma 4, we have $T_\alpha^C=\alpha^+(T_\alpha\backslash\{\bar{1}\})$. Conversely, assume that $T_\alpha^C=\alpha^+(T_\alpha\backslash\{\bar{1}\})$. If α is not a prefix set, then there exist $A_1,A_2\in\alpha\subseteq T_\alpha$ and $U\in M(X)\backslash\{\bar{1}\}$ such that $A_2=A_1U$. If $U\in T_\alpha$, then

$$A_2=A_1U\in\alpha(T_\alpha\backslash\{\bar{1}\})\subseteq\alpha^+(T_\alpha\backslash\{\bar{1}\})=T_\alpha^C$$

a contradiction. If $U\in T_\alpha^C$, then

$$A_2=A_1U\in\alpha T_\alpha^C=\alpha\alpha^+(T_\alpha\backslash\{\bar{1}\})\subseteq\alpha^+(T_\alpha\backslash\{\bar{1}\})=T_\alpha^C$$

a contradiction. Thus α is a prefix set. By Lemma 4, we have $T_\alpha^C=\alpha^+(T_\alpha\backslash\{\bar{1}\})=\alpha^+T_\alpha\backslash\alpha$. Then, by Theorem 4, α is a maximal prefix set.

Let

$$D(M(X))=\{\alpha\subseteq M(X)\mid \alpha\beta=\alpha\gamma \text{ implies } \beta=\gamma \text{ for all } \beta,\gamma\subseteq M(X)\}$$

Then $D(M(X))$ with the concatenation operation is a monoid with the identity $\{\bar{1}\}$. We call $D(M(X))$ the left cancellative semigroup.

Remark 5　Let α be a maximal prefix set, then by Theorem 2 and Theorem 5, $\alpha(M(X)\backslash\{\bar{1}\})=\alpha^{+}(T_{\alpha}\backslash\{\bar{1}\})$. Let $X=\{a, b\}$ and $\alpha=M(X)\backslash\{\bar{1}\}$, then $\alpha=\alpha^{+}=M(X)\backslash\{\bar{1}\}$ and $T_{\alpha}=M(X)$. Thus $\alpha(M(X)\backslash\{\bar{1}\})=\alpha^{+}(T_{\alpha}\backslash\{\bar{1}\})$. Since $\{a^{2}\}=\{a\}\{a\}\in\alpha\cap\alpha(M(X)\backslash\{\bar{1}\})$, then α is not a prefix set and so α is not a maximal prefix set. This example shows that if $\alpha(M(X)\backslash\{\bar{1}\})=\alpha^{+}(T_{\alpha}\backslash\{\bar{1}\})$, then α need not to be maximal prefix set.

However, we have:

Theorem 6　Let α be a prefix set and $\alpha\in D(M(X))$, then α is a maximal prefix set if and only if $\alpha(M(X)\backslash\{\bar{1}\})=\alpha^{+}(T_{\alpha}\backslash\{\bar{1}\})$.

Proof　Let α be a maximal prefix set. By Theorem 2 and Theorem 5, we have $\alpha(M(X)\backslash\{\bar{1}\})=\alpha^{+}(T_{\alpha}\backslash\{\bar{1}\})$. Conversely, assume that $\alpha(M(X)\backslash\{\bar{1}\})=\alpha^{+}(T_{\alpha}\backslash\{\bar{1}\})$, then $\alpha(M(X)\backslash\{\bar{1}\})=\alpha^{+}(T_{\alpha}\backslash\{\bar{1}\})=\alpha(\alpha^{*}(T_{\alpha}\backslash\{\bar{1}\}))$. Since $\alpha\subseteq D(M(X))$, we have $M(X)\backslash\{\bar{1}\}=\alpha^{*}(T_{\alpha}\backslash\{\bar{1}\})$. Thus

$$M(X)=(M(X)\backslash\{\bar{1}\})\cup\{\bar{1}\}=\alpha^{*}(T_{\alpha}\backslash\{\bar{1}\})\cup\{\bar{1}\}\subseteq\alpha^{*}T_{\alpha}$$

and so $M(X)=\alpha^{*}T_{\alpha}$. By Theorem 3, we have that α is a maximal prefix set.

Moreover, we have:

Theorem 7　Let $\alpha\in D(M(X))$ and $\bar{1}\notin\alpha$, then α is a maximal prefix set if and only if $\alpha(M(X)\backslash\{\bar{1}\})=\alpha^{+}T_{\alpha}\backslash\alpha$.

Proof　Let α be a maximal prefix set. By Theorem 2 and Theorem 4, we have $\alpha(M(X)\backslash\{\bar{1}\})=\alpha^{+}T_{\alpha}\backslash\alpha$. Conversely, assume that $\alpha(M(X)\backslash\{\bar{1}\})=\alpha^{+}T_{\alpha}\backslash\alpha$. Since $\alpha\cap\alpha(M(X)\backslash\{\bar{1}\})=\alpha\cap(\alpha^{+}T_{\alpha}\backslash\alpha)=\emptyset$, we have that α is a prefix set. By Lemma 4, we have $\alpha(M(X)\backslash\{\bar{1}\})=\alpha^{+}T_{\alpha}\backslash\alpha=\alpha^{+}(T_{\alpha}\backslash\{\bar{1}\})$. Then by Theorem 6, α is a maximal prefix set.

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极大前缀集的一些刻画

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摘要：设 $M(X)$是字母表 X 上的语言幺半群. 给出了 $M(X)$的极大前缀集的一些刻画.

关键词：前缀集；极大前缀集；左消去半群

责任编辑 廖 坤