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Characterization of $L_2(2^7)$ by $\tau_e(L_2(2^7))$ ①

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Abstract: Let G be a finite group and $\pi_e(G)$ the set of element orders of G. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. Let $\tau_e(G) = \{m_k \mid k \in \pi_e(G)\}$. In this paper, $L_2(2^7)$ is characterizable by $\tau_e(L_2(2^7))$, in other words, if G is a group such that $\tau_e(G) = \tau_e(L_2(2^7)) = \{1, 16\,383, 16\,256, 341\,376, 1\,040\,256, 682\,752\}$, then G is isomorphic to $L_2(2^7)$.

Key words: element order; characterizable; the number of elements with the same order

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we use P_r and n_r to denote a Sylow r-subgroup and the number of Sylow r-subgroups of group G, respectively. Let $k \in \pi_e(G)$, then we denote the number of elements of order k in G by m_k. Let $\tau_e(G)=\{m_k \mid k \in \pi_e(G)\}$. References [1-2] posed a very interesting problem related to algebraic number fields as follows:

Thompson Problem[1]　Let $\Gamma_1(G)=\{(n, S_n) \mid n \in \pi_e(G), S_n \in \tau_e(G)\}$, where S_n is the number of elements with order n. Suppose that $\Gamma(G)=\Gamma(H)$. If G is a finite solvable group, is it true that H is also necessarily solvable?

So far, no one can solve this problem completely even give a counterexample. We know that $\Gamma(G)$ consists of two sets, that is, $\pi_e(G)$ and $\tau_e(G)$. Reference [3] studied the case of the simple group A_5, and proved an interesting result using only $\pi_e(G)$, that is, a finite group G is isomorphic to A_5 if and only if $\pi_e(G)=\{1, 2, 3, 5\}$. Afterward, many simple groups are characterized using only the set of element orders and there are many relative papers. Of course, the following question is valuable: Considering the sizes of elements of same order but disregarding the actual orders of elements in $\Gamma_1(G)$ of Thompson Problem, in other words, it remains only $\tau_e(G)$, whether can it characterize finite simple groups? In other words, let G be a finite simple group, whether can it be characterized using only the set $\tau_e(G)$?

We denote by $k(\tau_e(G))$ the number of isomorphism classes of finite groups H satisfying $\tau_e(G) = \tau_e(H)$. By using this function, we pose the following definition:

Definition 1　Given a natural number n, a finite group G is called n-recognizable by $\tau_e(G)$ if $k(\tau_e(G)) = n$. Usually an 1-recognizable group is called a characterizable group. If there exist infinitely many non-isomorphic finite groups H such that $\tau_e(G)=\tau_e(H)$, then we call G a non-recognizable group by $\tau_e(G)$.

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In reference [4], it is proved that A_5 is determined by $\tau_e(A_5)$. In reference [5], it is shown that if G is a group and $\tau_e(G)=\tau_e(PSL(2, q))$, where $q \in \{7, 8, 11, 13\}$, then $G \cong PSL(2, q)$. In fact references [4－5] proved that some simple groups can be determined by $\tau_e(G)$ when $|\tau_e(G)|$ is smaller than 6. Is it true that G can be characterized by $\tau_e(G)$ if G is a finite simple group and $|\tau_e(G)| \geqslant 6$? In this paper, we continue this work and show that $L_2(2^7)$ is characterizable by $\tau_e(L_2(2^7))$.

Before starting the proof of the main result, we will mention a well-known result of Frobenius (see reference [6]), which is quoted frequently in the sequel.

Lemma 1[6]　Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G)=\{g \in G \mid g^m=1\}$, then $m \mid |L_m(G)|$.

Lemma 2[4]　Let G be a group containing more than two elements. If the maximal number s of elements of the same order in G is finite, then G is finite and $|G| \leqslant s(s^2-1)$.

Lemma 3[7]　If G be a finite group with $t(G)>1$, then one of the following statements holds:

(a) $t(G)=2$ and G is a Frobenius group;

(b) $t(G)=2$ and G is a 2-Frobenius group, i. e., $G=ABC$, where $A \lhd G$, $AB \lhd G$, AB is a Frobenius group with kernel A and complement B, and BC is also a Frobenius group with kernel B and complement C, and G is solvable;

(c) there exists a nonabelian simple group S such that $S \leqslant \bar{G}=G/K \leqslant \mathrm{Aut}(S)$, where K is the maximal normal soluble subgroup of G; furthermore, K and $\bar{G}/S$ are $\pi_1(G)$-groups, $\Gamma(S)$ is disconnected, $t(S) \geqslant t(G)$.

Theorem 1　Let G be a group such that

$$\tau_e(G)=\tau_e(L_2(2^7))=\{1, 16\,383, 16\,256, 341\,376, 1\,040\,256, 682\,752\}$$

then $G \cong L_2(2^7)$.

Proof　Let S_m be the number of elements of order m. By Lemma 2 we can assume that G is finite. Note that $S_m=k\varphi(m)$, where k is the number of cyclic subgroups of order m, and $\varphi(m)$ is Euler totient function. Moreover, if $m>2$, then $\varphi(m)$ is even.

First we claim that $\pi(G) \subseteq \{2, 3, 43, 127\}$. Since $16\,383 \in \tau_e(G)$, it follows that $2 \in \pi(G)$ and $S_2=16\,383$. Suppose that $p>2$ is a prime and $p \in \pi(G)$, then by Lemma 1 we have $p \mid (1+S_p)$ for some $S_p \in \{16\,256, 341\,376, 1\,040\,256, 682\,752\}$, and it follows that $\pi(G) \subseteq \{2, 3, 5\,419, 17, 43, 467, 127, 8\,191, 241, 2\,833\}$. Since $\varphi(p) \mid S_p$, we get that $5\,419, 467, 8\,191, 241, 2\,833 \notin \pi(G)$, and so $\pi(G) \subseteq \{2, 3, 17, 43, 127\}$. Suppose that $17 \in \pi(G)$, then $S_{17}=341\,376$ since $\varphi(17) \mid S_{17}$. Similarly, we get that $17^2 \notin \pi(G)$. Let P_{17} be a Sylow 17-subgroup of G. Then by Lemma 1, it follows that $|P_{17}| \mid (1+S_{17})$, namely, $|P_{17}| \mid 341\,377=17 \cdot 43 \cdot 467$, which implies that $|P_{17}|=17$. Hence $n_{17}=\dfrac{S_{17}}{\varphi(17)}=2^3 \cdot 3 \cdot 7 \cdot 127$. Consequently $7 \in \pi(G)$, which is a contradiction. Thus $17 \notin \pi(G)$. Therefore $\pi(G) \subseteq \{2, 3, 43, 127\}$. In addition, if $3 \in \pi(G)$, then $S_3=16\,256=2^7 \cdot 127$; if $43 \in \pi(G)$, then $S_{43}=341\,376=2^7 \cdot 3 \cdot 7 \cdot 127$; if $127 \in \pi(G)$, then $S_{127}=1\,040\,256=2^7 \cdot 3^3 \cdot 7 \cdot 43$.

Now we assume that $|G|=2\,097\,024+16\,256k_1+341\,376k_2+1\,040\,256k_3+682\,752k_4=2^a \cdot 3^b \cdot 43^c \cdot 127^d$, namely

$$2^7 \cdot 3 \cdot 43 \cdot 127+2^7 \cdot 127k_1+2^7 \cdot 3 \cdot 7 \cdot 127k_2+2^7 \cdot 3^3 \cdot 7 \cdot 43k_3+2^8 \cdot 3 \cdot 7 \cdot 127k_4=2^a \cdot 3^b \cdot 43^c \cdot 127^d \quad (1)$$

where a, b, c and d are non-negative integers. And we consider the following Cases:

Case 1　Let $\pi(G)=\{2\}$. If $2^i \in \pi_e(G)$, then $i \leqslant 9$ since $\varphi(2^i) \mid S_{2^i}$, thus $|\pi_e(G)| \leqslant 10$. From equation (1), it follows that $2^7 \cdot 3 \cdot 43 \cdot 127+2^7 \cdot 127k_1+2^7 \cdot 3 \cdot 7 \cdot 127k_2+2^7 \cdot 3^3 \cdot 7 \cdot 43k_3+2^8 \cdot 3 \cdot 7 \cdot 127k_4=2^a$, $0 \leqslant k_1+k_2+k_3+k_4 \leqslant 4$. Hence $2\,097\,024 \leqslant 2^a \leqslant 2\,097\,024+1\,040\,256 \times 4$, and it follows that $21 \leqslant a \leqslant 22$, so $127 \mid (3^3 \cdot 7 \cdot 43k_3-2^{a-7})$, namely, $127 \mid (2^{a-8}-63k_3)$. If $a=21$, then $127 \mid (2^{13}-63k_3)$, and so $127 \mid (64-63k_3)$, which is impossible since $0 \leqslant k_3 \leqslant 4$. If $a=21$, similarly we can get a contradiction.

Case 2 Let $\pi(G)=\{2, 3\}$. Then $|G|=2^a \cdot 3^b$ by equation (1). If $b=1$, then $n_3=\frac{S_3}{\varphi(3)}=2^6 \cdot 127$, and it follows that $127 \in \pi(G)$, which is a contradiction, therefore $b>1$. If $9 \notin \pi_e(G)$, then $3^b \mid (1+S_3)$ by Lemma 1, namely, $3^b \mid 3 \cdot 5\,419$, which is impossible since $b>1$. Thus $9 \in \pi_e(G)$ and $S_9=341\,376$ since $9 \mid (1+S_3+S_9)$ for $S_9 \in \{16\,256, 341\,376, 1\,040\,256, 682\,752\}$. If $27 \in \pi_e(G)$, then $S_{27}=1\,040\,256$ since $\varphi(27) \mid S_{27}$, and by Lemma 1 we have $27 \mid (1+S_3+S_9+S_{27})$, namely, $27 \mid 1\,397\,889$, which is a contradiction, hence $27 \notin \pi_e(G)$. Since $3^b \mid (1+S_3+S_9)$, it follows that $b=2$, and so $|G|=2^a \cdot 3^2$. If $2^i \in \pi_e(G)$, then $i \leqslant 9$ since $\varphi(2^i) \mid S_{2^i}$ for $S_{2^i} \in \{16\,383, 16\,256, 341\,376, 1\,040\,256, 682\,752\}$. Similarly, if $2^i \cdot 3^j \in \pi_e(G)$, then $i \leqslant 8$ and $j \leqslant 2$. Thus $|\pi_e(G)| \leqslant 28$. From equation (1) we get that $0 \leqslant k_1+k_2+k_3+k_4 \leqslant 22$ and $127 \mid (3 \cdot 7 \cdot 43k_3 - 2^{a-7})$, namely, $127 \mid (7k_3 - 2^{a-8})$. Similarly to Case 1, we can get that $18 \leqslant a \leqslant 21$ and $127 \nmid (2^{a-8}-7k_3)$ since $0 \leqslant k_3 \leqslant 22$. Now we get a contradiction.

Case 3 Let $\pi(G)=\{2, 43\}$. If $43^2 \in \pi_e(G)$, then $S_{43^2}=1\,040\,256$ since $\varphi(43^2) \mid S_{43^2}$. Thus by Lemma 1, it follows that $43^2 \mid (1+S_{43}+S_{43^2})$, namely, $43^2 \mid 1\,381\,633$, which is a contradiction. Therefore $43^2 \notin \pi_e(G)$. Let P_{43} be a Slylow 43-subgroup of G. Then $|P_{43}| \mid (1+S_{43})$, namely, $|P_{43}| \mid 341\,377$, and it follows that $|P_{43}|=43$. Hence $n_{43}=\frac{S_{43}}{\varphi(43)}=2^6 \cdot 127$, and so $127 \in \pi(G)$, which is a contradiction.

Similarly we can show that $\pi(G) \neq \{2, 127\}, \{2, 3, 43\}, \{2, 3, 127\}, \{2, 43, 127\}$.

Case 4 Let $\pi(G)=\{2, 3, 43, 127\}$, then $|G|=2^a \cdot 3^b \cdot 43^c \cdot 127^d$. Similarly to Case 3, we can prove that $|P_{43}|=43$, and so $c=1$. Also we get that $d=1$ in the same way. If $3 \cdot 127 \in \pi_e(G)$, then $S_{3\cdot 127}=1\,040\,256$ since $\varphi(3 \cdot 127) \mid S_{3\cdot 127}$, and it follows that $3 \cdot 127 \mid (1+S_3+S_{127}+S_{3\cdot 127})$ by Lemma 1, namely, $3 \cdot 127 \mid 2\,096\,769$, which is a contradiction, thus $3 \cdot 127 \notin \pi_e(G)$. Similarly, we get that $2 \cdot 127, 43 \cdot 127 \notin \pi_e(G)$. Now we consider a Sylow 3-subgroup P_3 of G acts point freely on the set of elements of order 127, then $3^b \mid S_{127}$, namely, $3^b \mid 2^7 \cdot 3^3 \cdot 7 \cdot 43$, and so $b \leqslant 3$. Similarly, we can show that $a \leqslant 7$. On the other hand, from equation (1) we have $2^7 \mid 2^a$, and so $a \geqslant 7$. Consequently $a=7$. Therefore $|G|=2^7 \cdot 3^b \cdot 43 \cdot 127 (1 \leqslant b \leqslant 3)$.

Suppose that $b=2$. It is easy to see that $t(G) \geqslant 2$ since $2 \cdot 127, 3 \cdot 127, 43 \cdot 127 \notin \pi_e(G)$. In the following we discuss three cases by Lemma 3.

(1°) G is not a Frobenius group. If not, then G is a Frobenius group with kernel A and complement B. By the basic properties of a Frobenius group we know that $(|A|, |B|)=1$ and $|B| \mid (|A|-1)$. Since $\{127\}$ is an odd component of $\Gamma(G)$, it follows that $|A|=2^7 \cdot 3^2 \cdot 43$ and $|B|=127$, or $|A|=127$ and $|B|=2^7 \cdot 3^2 \cdot 43$, which implies that $|B| \nmid (|A|-1)$, which is a contradiction.

(2°) G is not a 2-Frobenius group. It is not hard to see that G is not a 2-Frobenius group by Lemma 3 since $|G|=2^7 \cdot 3^2 \cdot 43 \cdot 127$.

(3°) By Lemma 3, G has a normal series: $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is simple. By references [8-9] and comparing the group order, we get that $K/H \cong L_2(2^7), L_2(127)$.

If $K/H \cong L_2(127)$, then $7 \in \pi(G)$, which is a contradiction.

If $K/H \cong L_2(2^7)$, then $|K|=2^7 \cdot 3 \cdot 43 \cdot 127$ or $|K|=2^7 \cdot 3^2 \cdot 43 \cdot 127$. If $|K|=2^7 \cdot 3 \cdot 43 \cdot 127$, then $H=1$ and $K \cong L_2(2^7)$. Since $2 \cdot 127, 3 \cdot 127, 43 \cdot 127 \notin \pi_e(G)$, we get that $C_G(K)=1$. Since $G/C_G(K) \lesssim \mathrm{Aut}(K)$ and $|\mathrm{Aut}(K)|=2^7 \cdot 3 \cdot 43 \cdot 127 \cdot 7$, it follows that $|G| \mid 2^7 \cdot 3 \cdot 43 \cdot 127 \cdot 7$, namely, $2^7 \cdot 3^2 \cdot 43 \cdot 127 \mid 2^7 \cdot 3 \cdot 43 \cdot 127 \cdot 7$, which is a contradiction. If $|K|=2^7 \cdot 3^2 \cdot 43 \cdot 127$, then $|H|=3$. Let P_{127} be a Sylow 127-subgroup of G. Since P_{127} acts fixed-point-freely on H, it follows that HP_{127} is a Frobenius group with kernel H and complement P_{127}. Thus $|P_{127}| \mid (|H|-1)$, namely, $127 \mid 2$, which is impossible.

By the same reason, we can show that $b \neq 3$. If $b=1$, then G is neither a Frobenius group nor a 2-Frobenius group. Then by Lemma 3, G has a normal series: $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is simple. By references [8-9] and comparing the group order, we get that $K/H \cong L_2(2^7), L_2(127)$. If $K/H \cong L_2(127)$, then $7 \in \pi(G)$, which is a contradiction; If $K/H \cong L_2(2^7)$, then $H=1$, $K=G$, and so $G \cong L_2(2^7)$. Now the proof of Theorem 1 is complete.

Remark 1 By Chap. 2 and Theorems 8.2—8.5 of reference [10], we can get the following statements:

(1°) If $2 \nmid q$, then $\tau_e(L_2(q)) = \{1, \frac{\varphi(d)\cdot q\cdot(q+1)}{2}, 1 < d \mid \frac{q-1}{2}, \frac{\varphi(s)\cdot q\cdot(q-1)}{2}, 1 < s \mid \frac{q+1}{2}, q^2-1\}$;

(2°) If $2 \mid q$, then $\tau_e(L_2(q)) = \{1, \frac{\varphi(d)\cdot q\cdot(q+1)}{2}, 1 < d \mid (q-1), \frac{\varphi(s)\cdot q\cdot(q-1)}{2}, 1 < s \mid (q+1), q^2-1\}$, where φ is Euler's totient function.

Problem 1 We try to make a further study to the problem of characterization of finite simple groups by $\tau_e(G)$, thus we give Remark 1. Now from references [4—5], we know that $L_2(2^n)$ can be characterized by $\tau_e(L_2(2^n))(n=2,3,7)$. Is it true that $L_2(2^m)$ can be characterized by $\tau_e(L_2(2^m))$ for an arbitrary natural number m?

Problem 2 Let G be a finite simple group, then from Lemma 2 we know that G is n-recognizable by $\tau_e(G)$ for some natural number n. Do there exist two finite simple groups G and H not isomorphic to each other such that $\tau_e(G)=\tau_e(H)$?

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用 $\tau_e(L_2(2^7))$ 刻画 $L_2(2^7)$

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摘要：令 G 为有限群，$\pi_e(G)$ 为 G 的元素的阶的集合，$k \in \pi_e(G)$，m_k 表示 G 中 k 阶元的个数，$\tau_e(G)=\{m_k \mid k \in \pi_e(G)\}$. 证明 $L_2(2^7)$ 可用 $\tau_e(L_2(2^7))$ 加以刻画，换言之，当 G 为群且满足 $\tau_e(G)=\tau_e(L_2(2^7))=\{1, 16\ 383, 16\ 256, 341\ 376, 1\ 040\ 256, 682\ 752\}$ 时，有 $G \cong L_2(2^7)$.

关键词：元素的阶；可刻画的；同阶元长度

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