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The Vulnerability of Cartesian Products of Graphs①

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Abstract: In order to describe the vulnerability of some useful networks and offer valuable methods for network designers to construct efficient networks, the method of this paper is to measure the vulnerability of these networks by determining the rupture degree of Cartesian product of complete graphs and grids.

Key words: the rupture degree; Cartesian product; grids

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Here we consider only undirected simple graphs. A good reference for any undefined terms is reference [1]. Let G be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. A set $X\subset V(G)$ is a cut-set of G, if $G-X$ is disconnected. We denote the number of components and the order of a largest component in $G-X$ by $\omega(G-X)$ and $m(G-X)$, respectively. As usual, denote the path and complete graph on n vertices by P_n and K_n, respectively. The rupture degree of G is

$$r(G)=\max\{\omega(G-X)-|X|-m(G-X):X\in V(G),\ \omega(G-X)>1\}$$

The Cartesian product $G_1\times G_2\times\cdots\times G_k$ of graphs $G_1, G_2, \cdots, G_k$ is a graph with vertex set $V(G_1)\times V(G_2)\times\cdots\times V(G_k)$ and two vertices $u=(u_1, u_2, \cdots, u_k)$ and $v=(v_1, v_2, \cdots, v_k)$ adjacent iff for exactly one i, $u_i\neq v_i$ and (u_i, v_i) is an edge in G_i. In this paper, the rupture degree of Cartesian product $K_m\times K_n$, hypercubes Q_k and grids $P_{n_1}\times P_{n_2}\times\cdots\times P_{n_k}$ are determined.

1 The rupture degree of $K_m\times K_n$

Defnition 1 A cut-set $X\subset V(G)$ is called r-set of G, if $\omega(G-X)-|X|-m(G-X)=r(G)$.

Theorem 1 If $1\leqslant m\leqslant n$, then $r(K_m\times K_n)=m+n-mn-\lceil\frac{n}{m}\rceil$.

Proof Denote

$$V(K_m)=\{u_1, u_2, \cdots, u_m\} \qquad V(K_n)=\{v_1, v_2, \cdots, v_n\}$$

then $V(K_m\times K_n)=\{(u_i, v_j): 1\leqslant i\leqslant m, 1\leqslant j\leqslant n\}$. For notational convenience, let $G=K_m\times K_n$. First, let F be the family of all r-sets A in G with the maximum number of components in $G-A$. Let X be an ele-

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ment of F with minimum order. Then we will show that the components of $G-X$ satisfy the following properties (a)—(d) and complete the proof:

(a) Every component C_i ($i=1,2,\cdots,\omega$) of $G-X$ is of form $K_{m_i}\times K_{n_i}$;

(b) Given any $u_i\in\{u_1, u_2, \cdots, u_m\}$ (or $v_j\in\{v_1, v_2, \cdots, v_n\}$), there exists a component C_i (or C_j) containing some (u_i, v_p) (respectively, (u_p, v_j)), where $v_p\in\{v_1, v_2, \cdots, v_n\}$ (respectively, $u_p\in\{u_1, u_2, \cdots, u_m\}$);

(c) For every component $K_{m_i}\times K_{n_i}$, either $m_i=1$ or $n_i=1$;

(d) There is no component $K_{m_i}\times K_{n_i}$ with $m_i>1$ and $n_i=1$.

Thus, all the components are of the form $K_1\times K_{n_i}$

$$\omega(G-X)=m \qquad |X|=mn-nm(G-X)\geqslant\lceil\frac{n}{m}\rceil$$

Hence, $r(K_m\times K_n)=m+n-mn-\lceil\frac{n}{m}\rceil$.

Proof of property (a) It is enough if we show that whenever $(u_i, v_r),(u_i, v_s),(u_j, v_r)\in V(C_k)$, then $(u_j, v_s)\in V(C_k)$, $(u_j, v_s)\notin V(C_k)$. On the contrary, suppose that $(u_j, v_s)\in V(G-C_k)$. By the definition of $K_{m_i}\times K_{n_i}$, $(u_j, v_s)\in X$. Let $X'=X-(u_j, v_s)$. Then (u_j, v_s) is adjacent with (u_i, v_s) and (u_j, v_r) in $G-X'$, and $[C_k\cup(u_j, v_s)]$ is a component of $G-X'$, so

$$|X'|=|X|-1 \qquad \omega(G-X')=\omega(G-X) \qquad m(G-X')\leqslant m(G-X)+1$$

Hence

$$\omega(G-X')-|X'|-m(G-X')\geqslant\omega(G-X)-|X|-m(G-X)$$

so X' is an r-set. But this contradicts the fact that X is an r-set and $|X|$ is of minimum order.

Proof of property (b) Assume the contrary, so for some i, we have

$$\{(u_i, v_j): 1\leqslant j\leqslant n\}\subseteq X \qquad 1\leqslant i\leqslant m$$

Let C be a component of $G-X$, and (u_k, v_r) be an element of C. Let $X'=X-(u_i, v_r)$. Then $[C\cup(u_i, v_r)]$ is a component in $G-X'$

$$|X'|=|X|-1 \qquad \omega(G-X')=\omega(G-X) \qquad m(G-X')\leqslant m(G-X)+1$$

Hence, X' is also an r-set, this contradicts the fact that X is an r-set and $|X|$ is of minimum order.

Proof of property (c) Assume that there is a component $C_i=K_{m_i}\times K_{n_i}$ with $m_i>1$ and $n_i>1$. For notational convenience, let $m_i=s$, $n_i=t$, $C_i=C_1$. Without loss of generality, assume that $C_1=K_s\times K_t$, where

$$V(K_s)=\{u_1, u_2, \cdots, u_s\} \qquad V(K_t)=\{v_1, v_2, \cdots, v_t\}$$

Let

$$X'=X\cup\{(u_s, v_1), (u_s, v_2), \cdots, (u_s, v_{t-1})\}\cup\{(u_1, v_t), (u_2, v_t), \cdots, (u_{s-1}, v_t)\}$$

Then

$$|X'|=|X|+s+t-2 \qquad \omega(G-X')=\omega(G-X)+1 \qquad m(G-X')\leqslant m(G-X)$$

We now estimate the size of $|X|$ and $m(G-X)$. Since C_1 is a component in $G-X$, and

$$X\in\{(u_i, v_j): 1\leqslant i\leqslant s, t+1\leqslant j\leqslant n; s+1\leqslant i\leqslant m, 1\leqslant j\leqslant t\}$$

so

$$|X|\geqslant s(n-t)+t(m-s)\geqslant s[\omega(G-X)-1]+t[\omega(G-X)-1]=[\omega(G-X)-1](s+t)$$

On the other hand

$$m(G-X)\geqslant|C_1|=st\geqslant s+t-2\geqslant s+t-2\omega(G-X)$$

So we obtain

$$\frac{|X'|+m(G-X')}{\omega(G-X')} \leqslant \frac{|X|+s+t-2+m(G-X)}{\omega(G-X)+1} \leqslant \frac{|X|+m(G-X)}{\omega(G-X)}$$

$$|X'|+m(G-X') \leqslant \frac{\omega(G-X')[|X|+m(G-X)]}{\omega(G-X)}$$

Consider the value of

$$\omega(G-X')-|X'|-m(G-X')$$

It is easy to have

$$\omega(G-X')-|X'|-m(G-X') \geqslant \omega(G-X') - \frac{\omega(G-X')[|X|+m(G-X)]}{\omega(G-X)} \geqslant$$

$$\omega(G-X)-|X|-m(G-X)$$

Hence, X' is an r-set of G with $\omega(G-X')>\omega(G-X)$, which is a contradiction to the choice of X.

Proof of property (d) It is clear that if there is a component $C_i=K_{m_i}\times K_1$ with $m_i>1$, then there is a component $C_j=K_1\times K_{n_j}$ with $n_j>1$; otherwise, $n_j=1$ for every j, we immediately have a contradiction:

$$\omega(G-X) < \sum_{i=1}^{\omega} m_i = m \leqslant n = \sum_{j=1}^{\omega} n_j = \omega(G-X)$$

Next we prove property (d). Assume that there is a component $C=K_s\times K_1$ with $s>1$, then there must be a component $C'=K_1\times K_t$ with $t>1$. Now assume that

$$V(C)=\{(u_1, v_\alpha), (u_2, v_\alpha), \cdots, (u_s, v_\alpha)\}$$

$$V(C')=\{(u_\beta, v_1), (u_\beta, v_2), \cdots, (u_\beta, v_t)\}$$

where the set $\{u_1, u_2, \cdots, u_s\}$ and $\{v_1, v_2, \cdots, v_t\}$ don't contain u_β and v_α respectively. Let

$$X'=X\cup\{(u_s, v_\alpha)\}\cup\{(u_\beta, v_t)\}-\{(u_s, v_t)\}$$

Then

$$|X'|=|X|+1 \qquad \omega(G-X')=\omega(G-X)+1 \qquad m(G-X')\leqslant m(G-X)$$

Hence

$$\omega(G-X')-|X'|-m(G-X') \geqslant \omega(G-X)-|X|-m(G-X)$$

This means that X' is an r-set of G and $\omega(G-X')>\omega(G-X)$, which is a contradiction to the choice of X.

2 The rupture degree of grids

In this section, we consider the rupture degree of $P_{n_1}\times P_{n_2}\times\cdots\times P_{n_k}$. Firstly, we need the following Lemmas, which are obtained in reference [2].

Lemma 1[2] If graph G contains a Hamilton path, then so does $G\times P_n$.

Lemma 2[2] If $n\geqslant 1$, then $r(P_n)=0$ (n is an odd number), $r(P_n)=-1$ (n is an even number).

Lemma 3[2] If H is a spanning subgraph of G, then $r(G)\leqslant r(H)$.

Lemma 4[2] If $m\geqslant n$, then $r(K_{m,n})=m-n-1$.

Theorem 2 For positive integers $n_1, n_2, \cdots, n_k$, $r(P_{n_1}\times P_{n_2}\times\cdots\times P_{n_k})=0$ (all n_i are odd), $r(P_{n_1}\times P_{n_2}\times\cdots\times P_{n_k})=-1$ (some n_i is even).

Proof By Lemma 1, it follows that $P_{n_1}\times P_{n_2}\times\cdots\times P_{n_k}$ contains a Hamilton path $P_{n_1 n_2\cdots n_k}$. So by Lemmas 2 and 3 we know

$$r(P_{n_1}\times P_{n_2}\times\cdots\times P_{n_k}) \leqslant r(P_{n_1 n_2\cdots n_k}) = \begin{cases} 0 & \text{all } n_i \text{ are odd} \\ -1 & \text{some } n_i \text{ is even} \end{cases}$$

On the other hand, we know that if G is a bipartite graph with bipartition $[A, B]$, and H is a bipartite graph with bipartition $[C, D]$, then $G\times H$ is a bipartite graph with bipartition

$$[(A\times C)\cup(B\times D),(A\times D)\cup(B\times C)]$$

Therefore, we have that if all n_i are odd, then

$$P_{n_1}\times P_{n_2}\times\cdots\times P_{n_k}\subseteq K_{\frac{n_1n_2\cdots n_k-1}{2},\frac{n_1n_2\cdots n_k-1}{2}}$$

if some n_i is even, then

$$P_{n_1}\times P_{n_2}\times\cdots\times P_{n_k}\subseteq K_{\frac{n_1n_2\cdots n_k}{2},\frac{n_1n_2\cdots n_k}{2}}$$

Hence

$$r(P_{n_1}\times P_{n_2}\times\cdots\times P_{n_k})\geqslant\begin{cases}0 & \text{all } n_i \text{ are odd;}\\ -1 & \text{some } n_i \text{ is even}\end{cases}$$

which completes the proof.

Since Q_k is isomorphic to the Cartesian product of k copies of P_2, hence we have:

Corollary 1 The rupture degree of the hypercube $r(Q_k)=-1$ $(k\geqslant 2)$.

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卡氏积图的抗毁性

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摘要:为了描述一些常用网络结构的抗毁性，为网络设计者设计高效网络结构提供有价值的方法和依据，通过界定完全图的卡氏积和网格图等一些常用网络图的毁裂度来刻画它们的抗毁性.

关键词: 毁裂度；卡氏积；网格图

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