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Finite Groups Having at Most Eight Non-subnormal Subgroups①

FENG Ai-fang, LIU Zu-hua

Department of Mathematics, Kunming University, Kunming 650214, China

Abstract: $N(G)$ denotes the number of non-subnormal subgroups of G. The influences of $N(G)$ on the structure and properties of G are discussed. By using the properties of non-nilpotent finite inner-Abel group and the method of classified discussion, the finite groups with $N(G)\leqslant 8$ are completely classified.

Key words: finite group; non-subnormal subgroup; conjugate; maximal subgroup

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In group theory, there are various results about how the structure of a finite group is related to its special subgroups. The structure of groups whose subgroups are all normal (the Dedekind groups) has been completely described in reference [1]. And the finite groups with one conjugate class of non-normal subgroups are classified in reference [2]. Moreover, reference [3] classified all finite groups with one conjugate class of non-subnormal subgroup, and reference [4] proved that finite groups with two conjugate classes of non-subnormal subgroups are solvable. In this paper, finite groups with eight or fewer non-subnormal subgroups are completely classified.

Let G be a finite group. $\mu(G)$ denotes the number of conjugate classes of non-subnormal subgroups of G. $N(G)$ denotes the number of non-subnormal subgroups of G. $|\pi(G)|$ denotes the number of all prime divisors of $|G|$. $\Phi(G)$ denotes the Frattini subgroup of G. $A\ltimes B$ denotes the semidirect product of A and B with $B\lhd AB$. The rest of notations are referred to reference [1].

1 Preliminaries

Lemma 1[3] Let G be a finite group. Then $\mu(G)=1$ if and only if G is a finite non-nilpotent inner-abelian group, that is

$$G\cong P\ltimes Q=\langle a,\ b_1,\ b_2,\ \cdots,\ b_\beta \mid a^{p^\alpha}=1=b_1^q=b_2^q=\cdots=b_\beta^q;$$
$$[b_i,\ b_j]=1,\ i,j=1,2,\cdots,\beta;\ b_i^a=b_{i+1},\ i=1,2,\cdots,\beta-1;\ b_\beta^a=b_1^{d_1}b_2^{d_2}\cdots b_\beta^{d_\beta}\rangle$$

where $f(x)=x^\beta-d_\beta x^{\beta-1}-\cdots-d_2x-d_1$ is an irreducible polynomial over the field F_q, which divides x^p-1, and $q^\beta\equiv 1(\bmod p)$. In particular, if $N(G)=q$, where q is a prime, then $\mu(G)=1$ if and only if

$$G\cong P\ltimes Q=\langle a,\ b \mid a^{p^\alpha}=1=b^q,\ b^a=b^r\rangle$$

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作者简介:冯爱芳(1981-),女,山东济宁人,讲师,主要从事无限群的研究.

where $r \not\equiv 1(\bmod q)$, $r^p \equiv 1(\bmod q)$, and $q \equiv 1(\bmod p)$.

Lemma 2 Let G be a finite group, and C be a conjugate class of non-subnormal subgroups of G. Then $|C| \geqslant 3$.

proof Let $H \in C$. If $|C|=1$, then $H \lhd G$, a contradiction. If $|C|=2$, then $|G: N_G(H)|=2$, hence $H \lhd N_G(H) \lhd G$, which is a contradiction. So $|C| \geqslant 3$.

Lemma 3 Let G be a finite group with $N(G) \leqslant 8$. Then $\mu(G) \leqslant 2$. In particular, $\mu(G)=1$ if $N(G) \leqslant 5$.

proof Let C be a conjugate class of non-subnormal subgroups of G. Then $|C| \geqslant 3$ from Lemma 2. If $\mu(G) \geqslant 3$, then $N(G) \geqslant 3 \times 3 = 9$, a contradiction.

Lemma 4 Let G be a finite group with $\mu(G)=1$ and $N(G)=n$. Then $n=q^\beta$, where $\beta \geqslant 1$, and q is a prime.

proof By Lemma 1, $|G|=p^\alpha q^\beta$, where $\alpha, \beta \geqslant 1$, p, q are different primes. Without loss of generality, let $P \in \mathrm{Syl}_p(G)$ be non-subnormal. Then $P=N_G(P)$ and $n=|G: N_G(P)|=|G: P|=q^\beta$.

Lemma 5[4] Let G be a finite group with $\mu(G)=2$. Then

(i) G is solvable, and $|\pi(G)|=2,3$;

(ii) If H, K are non-subnormal and not conjugate in G, then there exists some $g \in G$ such that $H < K^g$. Moreover, H and K are maximal in K^g and G respectively, H is cyclic and $K=N_G(K)$.

2 Main results

Let G be a finite group with $N(G) \leqslant 8$. Then $3 \leqslant N(G) \leqslant 8$ by Lemma 2. Clearly, there exists some Sylow p-subgroup P such that $P \ntriangleleft G$. By Lemma 3 and Lemma 5, all Sylow subgroups of G except Sylow p-subgroups are normal in G.

Theorem 1 Let G be a finite group. Then

(i) If $N(G)=3$, then $G \cong \langle a, b \mid a^{2^\alpha}=1=b^3, b^a=b^{-1}\rangle$;

(ii) If $N(G)=4$, then $G \cong \langle a, b_1, b_2 \mid a^{3^\alpha}=1=b_1^2=b_2^2, [b_1, b_2]=1, (b_1)^a=b_2, (b_2)^a=b_1b_2\rangle$;

(iii) If $N(G)=5$, then $G \cong \langle a, b \mid a^{2^\alpha}=1=b^5, b^a=b^{-1}\rangle$.

proof If $N(G)=3,4,5$, then $\mu(G)=1$ by Lemma 3. By Lemma 1, assume that $|G|=p^\alpha q^\beta$, where p and q are different primes, $\alpha, \beta \geqslant 1$, $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$, and $P \ntriangleleft G$, $Q \lhd G$.

(i) If $N(G)=3$, according to Lemma 1, $G \cong \langle a, b \mid a^{p^\alpha}=1=b^3, b^a=b^r\rangle$, where $r \not\equiv 1(\bmod 3)$, $r^p \equiv 1(\bmod 3)$, and $3 \equiv 1(\bmod p)$. It follows that $p=2$, and $r=-1$.

(ii) If $N(G)=4$, according to Lemma 1, $q^\beta=4$, which implies that $q=2$, and $\beta=2$. Hence

$$G \cong P \ltimes Q = \langle a, b_1, b_2 \mid a^{p^\alpha}=1=b_1^2=b_2^2, [b_1, b_2]=1, (b_1)^a=b_2, (b_2)^a=(b_1)^{d_1}(b_2)^{d_2}\rangle$$

where $d_1, d_2=0,1$. By Sylow Theorem, $4 \equiv 1(\bmod p)$, and hence $p=3$. Since $\langle a^3\rangle \leqslant Z(G)$ and

$$(b_1)^{a^3}=(b_2)^{a^2}=((b_1)^{d_1}(b_2)^{d_2})^a=(b_2)^{d_1}((b_1)^{d_1}(b_2)^{d_2})^{d_2}=(b_1)^{d_1d_2}(b_2)^{d_1+d_2^2}=b_1$$

hence $d_1d_2 \equiv 1(\bmod 2)$, $d_1+(d_2)^2 \equiv 0(\bmod 2)$, which implies that $d_1=d_2=1$.

(iii) If $N(G)=5$, similar with the proof of Case (i), we have $p=2$, and $r=-1$. Hence Case (iii) in Theorem 1 holds.

Theorem 2 Let G be a finite group with $N(G)=6$. Then one of the following cases holds:

(i) $G \cong \langle a, b, c \mid a^{2^\alpha}=1=b^3=c^3, [b, a]=[b, c]=1, c^a=c^{-1}\rangle$.

(ii) $G \cong \langle a, b, c \mid a^{2^\alpha}=1=b^q=c^3, [b, a]=[b, c]=1, c^a=c^{-1}\rangle$, where $q \neq 2,3$.

proof If $N(G)=6$, then $\mu(G)=2$ by Lemma 3 and Lemma 4. Let C_1, C_2 be two conjugate classes of non-subnormal subgroups of G. Then $|C_1| \geqslant 3$, and $|C_2| \geqslant 3$ according to Lemma 2, hence $|C_1|=|C_2|=3$ since $|C_1|+|C_2|=6$. Let $H \in C_1$, $K \in C_2$, and without loss of generality, assume that $H < K$

and $K = N_G(K)$ by Lemma 5. We assert that $N_G(H) = K = N_G(K)$. In fact, if $N_G(H) \neq K$, then $N_G(H) = H$ by $\mu(G) = 2$. So

$$|C_1| = |G : N_G(H)| = |G : H| > |G : K| = |G : N_G(K)| = |C_2|$$

which contradicts $|C_1| = |C_2| = 3$. By Lemma 5 we have the following two cases:

(i) $|\pi(G)| = 2$. Let $|G| = p^\alpha q^\beta$, where p and q be different primes, $\alpha, \beta \geqslant 1$, $P \in Syl_p(G)$, $Q \in Syl_q(G)$, and $P \ntrianglelefteq G$, $Q \lhd G$.

Let P be conjugate to K. Then P is a 2-group with a cyclic maximal subgroup. By Sylow Theorem, $|C_2| = |G : N_G(K)| = |G : K| = q^\beta = 3$, that is, $q = 3$, $\beta = 1$, $Q = \langle c \mid c^3 = 1 \rangle$. Without loss of generality, let $H < P$. Then H is cyclic and maximal in P. Suppose that P is of type (i) of Theorem 5.14 in reference [5], that is, $P = \langle a \rangle$, $a^{2^\alpha} = 1$, $\alpha > 1$. Then $H = \langle b \mid b = a^2 \rangle$, and $c^a = c^{-1}$. However, $c^b = c^{a^2} = c$ implies that $H \lhd G$, a contradiction. Suppose that P is of types (ii), (iv), (v), (vi) or (vii) of Theorem 5.14 in reference [5]. Let $P = \langle a, b \mid a^{2^{\alpha-1}} = 1, b^2 = 1 \rangle$, and $H = \langle a \rangle$, then $c^a = c^{-1}$. Let $B = \langle b \rangle$, then B sn G by $\mu(G) = 2$, hence $B \lhd BQ = L$ since B is the Sylow 2-subgroup of L. Thus $L = B \times Q$, and $[b, c] = 1$. It is easily verified that $\langle a \rangle, \langle a^c \rangle, \langle a^{c^2} \rangle$ and $\langle a, b \rangle, \langle a^c, b \rangle, \langle a^{c^2}, b \rangle$ are 6 non-subnormal subgroups of G. However, $\langle ba \rangle$ or $\langle a^2, ba \rangle$ is also maximal in P. We assert that both $\langle ba \rangle$ and $\langle a^2, ba \rangle$ are non-subnormal in G. Otherwise, $P = \langle ba, b \rangle$ sn G, a contradiction. Moreover, $\langle ba \rangle \neq \langle a^c \rangle, \langle a^{c^2} \rangle$. Otherwise, without loss of generality, let $\langle ba \rangle = \langle a^c \rangle$, then $\langle ba \rangle \leqslant \langle a^c, b \rangle$, which follows that $a = (ab)b^{-1} \in \langle a^c, b \rangle$, thus $\langle a, b \rangle = \langle a^c, b \rangle$, a contradiction. Obviously, $\langle a^2, ba \rangle \neq \langle a \rangle, \langle a^c \rangle, \langle a^{c^2} \rangle$. Hence $N(G) > 6$, a contradiction.

Let P be conjugate to H. Then $|C_1| = 3 \equiv 1 \pmod p$ by Sylow Theorem, and therefor $p = 2$. By Lemma 5, G is solvable, P is cyclic. Without loss of generality, let $P < K$, then P and K are maximal in K and G respectively. Clearly K is also solvable, so K/P is a cyclic group with prime order q, and $|K| = 2^\alpha q$. Let $B = \langle b \mid b^q = 1 \rangle$ be a Sylow q-subgroup of K. Then B sn G by $\mu(G) = 2$, and then $B \lhd K$, since B is the Sylow subgroup of K, hence $K = P \times B$. By the maximality of K, $G = \langle K, L \rangle$, and $B \lhd \langle K, L \rangle = G$. Since $|C_2| = |G : K| = q^{\beta-1} = 3$, $q = 3$, $\beta = 2$, it follows that Q is an abelian group with order 9. If Q is cyclic, then $Q = \langle c \mid c^{3^2} = 1 \rangle$. Therefore, $G = \langle a, b, c \mid a^{2^\alpha} = c^{3^2} = 1, c^3 = b, [a, b] = 1, c^a = c^r \rangle$, where $r \not\equiv 1 \pmod 9$. Since $[a, b] = 1$, $b^a = (c^3)^a = (c^a)^3 = (c^r)^3 = c^{3r} = b = c^3$, hence $r \equiv 1 \pmod 3$. By $\mu(G) = 2$, $\langle a^2 \rangle$ sn G, and hence $\langle a^2 \rangle \lhd \langle a^2, c \rangle$ since $\langle a^2 \rangle$ is the Sylow subgroup of $\langle a^2, c \rangle$. Thus $\langle a^2, c \rangle = \langle a^2 \rangle \times \langle c \rangle$. It shows that $c^{a^2} = c^{r^2} = c$, and hence $r^2 \equiv 1 \pmod 9$, which contradicts $r \not\equiv 1 \pmod 9$ and $r \equiv 1 \pmod 3$. Thus Q is elementary abelian, and $Q = B \times \langle c \rangle$, where $|c| = 3$. For every Sylow 2-subgroup P of G, of course $P \cap B = 1$ has a complement in P, and clearly $\langle c \rangle$ is just the complement of B in Q. Therefore, B has a complement M in G by Corollary 4.7 in reference [5]. Now M satisfies $\mu(M) = 1$, and M is a non-nilpotent inner-abelian group. Clearly M is not conjugate to H and K. Thus $M \lhd G$ since $\mu(G) = 2$. According to Lemma 1, we have

$$G \cong B \times M = \langle a, b, c \mid a^{2^\alpha} = 1 = b^3 = c^3, [b, a] = [b, c] = 1, c^a = c^{-1} \rangle$$

where $M = \langle a, c \mid a^{2^\alpha} = 1 = c^3, c^a = c^{-1} \rangle$. Conversely, it is easily verified that the non-subnormal subgroups of G are $\langle a \rangle, \langle a^c \rangle, \langle a^{c^2} \rangle$, and $\langle a, b \rangle, \langle a^c, b \rangle, \langle a^{c^2}, b \rangle$. Hence $N(G) = 6$, and Case (i) in Theorem 2 holds.

(ii) $|\pi(G)| = 3$. Let $|G| = p^\alpha q^\beta r^\gamma$, where p, q and r be different primes, $\alpha, \beta, \gamma \geqslant 1$, $P \in Syl_p(G)$, $Q \in Syl_Q(G)$, $R \in Syl_r(G)$, and $P \ntrianglelefteq G$, Q, $R \lhd G$. We assert that P is conjugate to H. Otherwise, P is conjugate to K. By Lemma 5, K is maximal in G, hence $|G : K|$ is exponent of a prime since G is solvable. However, $|G : K| = |G : P| = q^\beta r^\gamma$, a contradiction. Let $P \lhd K$, then K/P is a cyclic group with prime order. Hence we assume that $K = P \times Q = \langle a, b \mid a^{p^\alpha} = b^q = [a, b] = 1 \rangle$. Then $|C_2| = |G : K| = r^\gamma = 3$, so $r = 3$, $\gamma = 1$. By Sylow Theorem, $3 \equiv 1 \pmod p$, which implies that $p = 2$. Let $M = P \ltimes R$. Then $\mu(M) = 1$, and M is a non-nilpotent inner-abelian group. By $\mu(G) = 2$, we have $M \lhd G$. So

$$G \cong M \times Q = \langle a, b, c \mid a^{2^\alpha} = 1 = b^q = c^3, [b, a] = [b, c] = 1, c^a = c^{-1} \rangle$$

Conversely, it is easily verified that the non-subnormal subgroups of G are $\langle a \rangle$, $\langle a^c \rangle$, $\langle a^{c^2} \rangle$, and $\langle a, b \rangle$, $\langle a^c, b \rangle$, $\langle a^{c^2}, b \rangle$. Hence $N(G) = 6$, and Case (ii) in Theorem 2 holds.

Theorem 3　Let G be a finite group with $N(G) = 7$. Then $G \cong \langle a, b \mid a^{p^\alpha} = 1 = b^7, b^a = b^r \rangle$, where $r \not\equiv 1 (\mathrm{mod}\ 7)$, $r^p \equiv 1(\mathrm{mod}\ 7)$, and $7 \equiv 1(\mathrm{mod}\ p)$.

proof　Assume that $N(G) = 7$. Now we assert that $\mu(G) = 1$. Otherwise, if C_1 and C_2 are two conjugate classes of non-subnormal subgroups of G, then $|C_1| + |C_2| = 7$. Thus $|C_1| = 3$, $|C_2| = 4$ or $|C_1| = 4$, $|C_2| = 3$ by Lemma 2. However, Lemma 5 implies that $|C_2| \,\Big|\, |C_1|$ or $|C_1| \,\Big|\, |C_2|$, which is a contradiction. So $G \cong \langle a, b \mid a^{p^\alpha} = 1 = b^7, b^a = b^r \rangle$, $r \not\equiv 1(\mathrm{mod}\ 7)$, $r^p \equiv 1(\mathrm{mod}\ 7)$, $7 \equiv 1(\mathrm{mod}\ p)$ according to Lemma 1.

Theorem 4　Let G be a finite group with $N(G) = 8$. Then one of the following cases holds:

(i) $G \cong \langle a, b_1, b_2, b_3 \mid a^{7^\alpha} = 1 = b_i^2, i = 1,2,3; [b_i, b_j] = 1, i,j = 1,2,3; b_1^a = b_2, b_2^a = b_3, b_3^a = b_1 b_2^r b_3^t \rangle$, where $r + t \equiv 1(\mathrm{mod}\ 2)$;

(ii) $G \cong \langle a, b, c_1, c_2 \mid a^{3^\alpha} = 1 = b^2 = c_1^2 = c_2^2, [b, a] = [b, c_1] = [b, c_2] = [c_1, c_2] = 1, c_1^a = c_2, c_2^a = c_1 c_2 \rangle$;

(iii) $G \cong \langle a, b, c \mid a^{3^m} = 1 = b^4, b^2 = c^2, b^c = b^3, b^a = c, c^a = bc \rangle$;

(iv) $G \cong \langle a, b, c_1, c_2 \mid a^{3^\alpha} = 1 = b^q = c_1^2 = c_2^2, [b, a] = [b, c_1] = [b, c_2] = [c_1, c_2] = 1, c_1^a = c_2, c_2^a = c_1 c_2 \rangle$, where $q \neq 2, 3$.

proof　Assume that $N(G) = 8$. Then $\mu(G) = 1, 2$ by Lemma 4 and Lemma 3.

(1) Suppose that $\mu(G) = 1$. By Lemma 1, Q is an elementary ablelian group with order 8. According to Sylow Theorem, $8 \equiv 1(\mathrm{mod}\ p)$, hence $p = 7$. So

$$G \cong P \ltimes Q = \langle a, b_1, b_2, b_3 \mid a^{7^\alpha} = 1 = b_1^2 = b_2^2 = b_3^2,$$
$$[b_i, b_j] = 1, i,j = 1,2,3, b_1^a = b_2, b_2^a = b_3, b_3^a = b_1^{d_1} b_2^{d_2} b_3^{d_3} \rangle$$

where $d_1, d_2, d_3 = 0, 1$. Since $\langle a^7 \rangle \leqslant Z(G)$, $b_1^{a^7} = b_1$, which implies that

$$2d_1^2 d_3 + d_1 d_2^2 + 3d_1 d_2 d_3^2 + d_1 d_3^4 \equiv 1(\mathrm{mod}\ 2)$$
$$d_1^2 + 4d_1 d_2 d_3 + d_1 d_3^3 + d_2^3 + 3d_2^2 d_3^2 + d_2 d_3^4 \equiv 0(\mathrm{mod}\ 2)$$
$$2d_1 d_2 + 3d_1 d_3^2 + 3d_2^2 d_3 + 4d_2 d_3^3 + d_3^5 \equiv 0(\mathrm{mod}\ 2)$$

The system of the equations has two solutions, that is, $d_1 = d_2 = 1$, $d_3 = 0$ and $d_1 = d_3 = 1$, $d_2 = 0$. So

$$G \cong P \ltimes Q = \langle a, b_1, b_2, b_3 \mid a^{7^\alpha} = 1 = b_1^2 = b_2^2 = b_3^2,$$
$$[b_i, b_j] = 1, i,j = 1,2,3, b_1^a = b_2, b_2^a = b_3, b_3^a = b_1 b_2^r b_3^t \rangle$$

where $r + t \equiv 1(\mathrm{mod}\ 2)$. Hence Case (i) in Theorem 4 holds.

(2) Suppose that $\mu(G) = 2$. Let C_1, C_2 be two conjugate classes of non-subnormal subgroups of G, $H \in C_1$, $K \in C_2$ and $H < K$, then $|C_2| \,\Big|\, |C_1|$. Hence $|C_1| = |C_2| = 4$ since $|C_i| \geqslant 3$ $(i = 1, 2)$. Similarly, we have $N_G(H) = K = N_G(K)$. By Lemma 5, $|\pi(G)| = 2$ or $|\pi(G)| = 3$.

When $|\pi(G)| = 2$, let $|G| = p^\alpha q^\beta$, where p and q are different primes, $\alpha, \beta \geqslant 1$, $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$, and $P \ntrianglelefteq G$, $Q \trianglelefteq G$.

Let P be conjugate to K. Then P is a 3-group with a cyclic maximal subgroup. By Sylow Theorem, $|C_2| = |G : N_G(K)| = |G : K| = q^\beta = 4$, that is, $q = 2$, $\beta = 2$, and Q is abelian. Let $H < P$ be a cyclic maximal subgroup of P. Then $M = H \ltimes Q$ is a non-nilpotent inner-abelian, and Q is an elementary abelian group. If P is of type (i) of Theorem 5.14 in reference [5], that is, $P = \langle a \rangle$, $a^{3^\alpha} = 1$, $\alpha > 1$, then

$$G \cong P \ltimes Q = \langle a, b, c_1, c_2 \mid a^{3^\alpha} = 1 = c_1^2 = c_2^2, b = a^3, c_1^b = c_2, c_2^b = c_1^{d_1} c_2^{d_2}, [c_1, c_2] = 1 \rangle$$

where $d_1, d_2 = 0, 1$. Since $\langle b^3 \rangle \leqslant Z(M)$, and

$$c_1^{b^3}=c_2^{b^2}=(c_1^{d_1}c_2^{d_2})^b=(c_2)^{d_1}(c_1^{d_1}c_2^{d_2})^{d_2}=c_1^{d_1d_2}c_2^{d_1+d_2^2}=c_1$$

It follows that $d_1d_2\equiv 1(\mathrm{mod}\ 2)$, $d_1+d_2^2\equiv 0(\mathrm{mod}\ 2)$. Hence $d_1=d_2=1$, that is $c_1^b=c_2$, $c_2^b=c_1c_2$. Let $c_1^a=c_1^{r_1}c_2^{r_2}$, $c_2^a=c_1^{s_1}c_2^{s_2}$. Then

$$c_1^b=c_1^{a^3}=(c_1^{r_1}c_2^{r_2})^{a^2}=((c_1^{r_1}c_2^{r_2})^{r_1}(c_1^{s_1}c_2^{s_2})^{r_2})^a=(c_1^{r_1^2+s_1r_2}c_2^{r_1r_2+s_2r_2})^a=$$

$$(c_1^{r_1}c_2^{r_2})^{r_1^2+s_1r_2}(c_1^{s_1}c_2^{s_2})^{r_1r_2+s_2r_2}=c_1^{r_1^3+2r_1r_2s_1+s_1s_2r_2}c_2^{r_1^2r_2+r_1r_2s_2+r_2^2s_1+r_2s_2^2}=c_2$$

So

$$r_1^3+2r_1r_2s_1+s_1s_2r_2\equiv 0(\mathrm{mod}\ 2)$$

$$r_1^2r_2+r_1r_2s_2+r_2^2s_1+r_2s_2^2\equiv 1(\mathrm{mod}\ 2)$$

The system of the equations has two solutions $r_1=0$, $r_2=1$, $s_1=0$, $s_2=1$ or $r_1=0$, $r_2=1$, $s_1=1$, $s_2=0$. However, if $r_1=0$, $r_2=1$, $s_1=0$, $s_2=1$, then $c_1^a=c_2$, $c_2^a=c_2$, hence $c_2^b=c_2^{a^3}=c_2$, a contradiction; if $r_1=0$, $r_2=1$, $s_1=1$, $s_2=0$, then $c_1^a=c_2$, $c_2^a=c_1$, which implies that $c_2^b=c_2^{a^3}=c_1^{a^2}=c_2^a=c_1$, a contradiction. Suppose that P is of type (ii) or (iii) of Theorem 5.14 in reference [5]. Similar with the proof of Theorem 2, we have $N(G)>8$, a contradiction.

Let P be conjugate to H. Then $|C_1|=4\equiv 1(\mathrm{mod}\ p)$ by Sylow Theorem, and therefore, $p=3$. By Lemma 5, G is solvable, $P=\langle a\mid a^{3^\alpha}=1\rangle$ is cyclic. Without loss of generality, let $P<K$. Then P and K are maximal in K and G respectively. Clearly, K is also solvable, so K/P is a cyclic group with prime order q, and $|K|=3^\alpha q$. Let $B=\langle b\mid b^q=1\rangle$ be a Sylow q-subgroup of K. Then B sn G by $\mu(G)=2$, which follows that $B\lhd K$, and hence $K=P\times B$. Since B sn Q, there exists some $L\leqslant Q$ such that $B\lhd L$ sn Q. Thus $G=\langle K, L\rangle$ since K is maximal in G, and hence $B\lhd\langle K, L\rangle=G$, which implies that $B\leqslant Z(Q)$ since $|B|=q$. Clearly, $\mu(G/B)=1$, therefore G/B is a non-nilpotent inner-abelian group, and Q/B is elementary abelian according to Lemma 1. Since $|C_2|=|G:K|=q^{\beta-1}=4$, $q=2$, $\beta=3$, then $|Q|=8$, and $Q\cong C_2\times C_2\times C_2$, $C_{2^2}\times C_2$, D_8, Q_8.

If $Q\cong C_2\times C_2\times C_2$, then $Q=B\times\langle c_1\rangle\times\langle c_2\rangle$, where $|C_1|=|C_2|=2$. For every Sylow 3-subgroup P of G, of course $P\cap B=1$ has a complement in P, and clearly $\langle c_1\rangle\times\langle c_2\rangle$ is just the complement of B in Q. Therefore, B has a complement M in G by Corollary 4.7 in reference [5]. Now M satisfies $\mu(M)=1$, and $M\lhd G$. By Lemma 1 we have

$$G\cong B\times M=\langle a, b, c_1, c_2\mid a^{3^\alpha}=1=b^2=c_1^2=c_2^2,$$

$$[b, a]=[b, c_1]=[b, c_2]=[c_1, c_2]=1, c_1^a=c_2, c_2^a=c_1^{d_1}c_2^{d_2}\rangle$$

where $d_1, d_2=0,1$. Since $\langle a^3\rangle\leqslant Z(M)$, and

$$c_1^{a^3}=c_2^{a^2}=(c_1^{d_1}c_2^{d_2})^a=(c_2)^{d_1}(c_1^{d_1}c_2^{d_2})^{d_2}=c_1^{d_1d_2}c_2^{d_1+d_2^2}=c_1$$

It follows that $d_1d_2\equiv 1(\mathrm{mod}\ 2)$, $d_1+d_2^2\equiv 0(\mathrm{mod}\ 2)$. Hence $d_1=d_2=1$. Conversely, it is easily verified that the non-subnormal subgroups of G are $\langle a\rangle$, $\langle a^{c_1}\rangle$, $\langle a^{c_2}\rangle$, $\langle a^{c_1c_2}\rangle$, and $\langle a, b\rangle$, $\langle a^{c_1}, b\rangle$, $\langle a^{c_2}, b\rangle$, $\langle a^{c_1c_2}, b\rangle$. Hence $N(G)=8$, and Case (ii) in Theorem 4 holds.

If $Q\cong C_{2^2}\times C_2$, then $Q=\langle c_1\rangle\times\langle c_2\rangle$, where $|c_1|=4$, $|c_2|=2$. Thus $B=\langle c_1^2\rangle$ since Q/B is an elementary abelian group. Let $\Omega_1(Q)=\{g\in Q\mid g^2=1\}$, then $\Omega_1(Q)=\langle c_1^2\rangle\times\langle c_2\rangle$, and $B\lhd\Omega_1(Q)<Q$. Clearly, $\Omega_1(Q)\lhd G$ since $\Omega_1(Q)$ Char $Q\lhd G$. It follows that $K=P\times B<P\times\Omega_1(Q)<P\ltimes Q=G$, which contradicts the maximality of K in G.

If $Q\cong D_8$, then $Q=\langle c, d\mid c^4=d^2=1, d^{-1}cd=c^3\rangle$, and $B=Z(Q)=\langle c^2\rangle$. Since there are only two elements c and c^3 with order 4, $c^a=c$ or $c^a=c^3$, then $\langle c\rangle\lhd G$, hence $K=P\times B<P\times\langle c\rangle<P\ltimes Q=G$, which contradicts the maximality of K in G.

If $Q\cong Q_8$, then $Q=\langle b, c\mid b^4=1, c^2=b^2, c^{-1}bc=b^3\rangle$, and $B=Z(Q)=\langle c^2\rangle$. Since G/B is a non-nilpotent inner-abelian group, $b^a=c$, $c^a=b^{r_1}c^{r_2}$, or $b^a=c^3$, $c^a=b^{r_1}c^{r_2}$, $r_1, r_2=1,3$. Thus

$$G \cong P \ltimes Q_8 = \langle a, b, c \mid a^{3^m} = 1 = b^4, b^2 = c^2, b^c = b^3, b^a = c, c^a = bc \rangle$$

Conversely, it is easily verified that the non-subnormal subgroups of G are $\langle a \rangle$, $\langle a^b \rangle$, $\langle a^c \rangle$, $\langle a^{bc} \rangle$, and $\langle a, b^2 \rangle$, $\langle a^b, b^2 \rangle$, $\langle a^c, b^2 \rangle$, $\langle a^{bc}, b^2 \rangle$. Hence $N(G) = 8$, and Case (iii) in Theorem 4 holds.

When $|\pi(G)| = 3$, let $|G| = p^\alpha q^\beta r^\gamma$, p, q and r be different primes, $\alpha, \beta, \gamma \geqslant 1$, $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_Q(G)$, $R \in \mathrm{Syl}_r(G)$, and $P \ntriangleleft G$, Q, $R \lhd G$. Similarly, P is conjugate to H. Without loss of generality, let $P \lhd K$, then K/P is a cyclic group with prime order. Hence we assume that $K = P \times Q = \langle a, b \mid a^{p^\alpha} = b^q = [a, b] = 1 \rangle$. Then $|C_2| = |G : K| = r^\gamma = 4$, so $r = 2$, $\gamma = 2$. By Sylow Theorem, $4 \equiv 1 (\bmod p)$, which implies that $p = 3$. Let $M = P \ltimes R$, then $\mu(M) = 1$, thus M is a non-nilpotent inner-abelian group. Since $\mu(G) = 2$, $M \lhd G$. By Lemma 1, we have

$$G \cong M \times Q = \langle a, b, c_1, c_2 \mid a^{3^\alpha} = 1 = b^q = c_1^2 = c_2^2,$$
$$[b, a] = [b, c_1] = [b, c_2] = [c_1, c_2] = 1, c_1^a = c_2, c_2^a = c_1^{d_1} c_2^{d_2}, q \neq 2, 3 \rangle$$

By $\langle a^2 \rangle \leqslant Z(M)$, we can get that $d_1 = d_2 = 1$. Conversely, it is easily verified that the non-subnormal subgroups of G are $\langle a \rangle$, $\langle a^{c_1} \rangle$, $\langle a^{c_2} \rangle$, $\langle a^{c_1 c_2} \rangle$, and $\langle a, b \rangle$, $\langle a^{c_1}, b \rangle$, $\langle a^{c_2}, b \rangle$, $\langle a^{c_1 c_2}, b \rangle$. Hence $N(G) = 8$, and Case (iv) in Theorem 4 holds.

According to Theorems 1, 2, 3, and 4, we obtain the classification of finite groups with $N(G) \leqslant 8$.

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至多含 8 个非次正规子群的有限群

冯爱芳，　刘祖华

昆明学院 数学系，昆明 650214

摘要: 令 $N(G)$ 为 G 中非次正规子群的个数. 讨论了 $N(G)$ 对群 G 的结构和性质的影响. 利用非幂零的有限内-Abel 群的性质和分类讨论的方法，对满足 $N(G) \leqslant 8$ 的有限群进行了完全分类.

关键词: 有限群；非次正规子群；共轭；极大子群

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