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两族 p^6 阶群的自同构群的阶[①]

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摘要：根据 p^6 阶群的分类，利用亚交换 p-群生成元的定义关系、自同构的性质以及数论中同余的一些运算性质，确定了两族 p^6 阶群 Φ_{25} 和 Φ_{26} 的自同构群的阶.

关　键　词：亚交换群；定义关系；自同构群；阶

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文献[1]对阶不超过 p^6 的 p-群进行了完全分类，已有一些学者确定了 p^6 阶群的自同构群的阶[2-8]. 本文给出了 p^6 阶群 Φ_{25} 和 Φ_{26} 的自同构群的阶.

在文中，p 总表示奇素数，$\alpha_{i+1}^{(p)}=\alpha_{i+1}^{p}\alpha_{i+2}^{\binom{p}{2}}\cdots\alpha_{i+k}^{\binom{p}{k}}\cdots\alpha_{i+p}$，$\nu$ 表示模 p 的最小的二次非剩余，其它的符号参见文献[9]. 同时，为了书写方便，将 $a\equiv b\ (\mathrm{mod}\ p)$ 记为 $a\equiv b$；如果 $ax\equiv 1\ (\mathrm{mod}\ p)$，则把 x 记为 a^{-1}，即 a^{-1} 表示 $aa^{-1}\equiv 1\ (\mathrm{mod}\ p)$.

1　预备引理

引理 1　设 G 是有限亚交换群，$G=\langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha]=\alpha_{i+1}, i=1,2,3\rangle$，则

(1°) $[\alpha_3^m, \alpha^{n_0}\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}\alpha_4^{n_4}]=\alpha_4^{mn_0}$；

(2°) $[\alpha_2^m, \alpha^{n_0}\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}\alpha_4^{n_4}]=\alpha_3^{mn_0}\alpha_4^{m\binom{n_0}{2}}$；

(3°) $[\alpha^{m_0}\alpha_1^{m_1}, \alpha^{n_0}\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}\alpha_4^{n_4}]=\alpha_2^{-m_0n_1+m_1n_0}\alpha_3^{-\binom{m_0}{2}n_1-m_0n_2+m_1\binom{n_0}{2}}\alpha_4^{-\binom{m_0}{3}n_1-\binom{m_0}{2}n_2-m_0n_3+m_1\binom{n_0}{3}}$.

证　(1°) 因为 G 是有限亚交换群，由文献[9]的第一章第四节的习题 9 及第十章的定理 3.1，我们可以得到

$$[\alpha_3^m, \alpha^{n_0}\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}\alpha_4^{n_4}]=[\alpha_3^m, \alpha^{n_0}\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}][\alpha_3^m, \alpha_4^{n_4}]=[\alpha_3^m, \alpha^{n_0}\alpha_1^{n_1}][\alpha_3^m, \alpha_2^{n_2}]=$$
$$[\alpha_3^m, \alpha_1^{n_1}][\alpha_3^m, \alpha^{n_0}]^{\alpha_1^{n_1}}=\alpha_4^{mn_0}$$

(2°) 易知

$$[\alpha_2^m, \alpha^{n_0}\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}\alpha_4^{n_4}]=[\alpha_2^m, \alpha^{n_0}\alpha_1^{n_1}]=[\alpha_2^m, \alpha_1^{n_1}][\alpha_2^m, \alpha^{n_0}]^{\alpha_1^{n_1}}=[\alpha_2^m, \alpha^{n_0}]^{\alpha_1^{n_1}}$$

由于

$$[\alpha_2^m, \alpha^{n_0}]=[\alpha_2, \alpha]^{\binom{m}{1}\binom{n_0}{1}}[\alpha_2, \alpha, \alpha]^{\binom{m}{1}\binom{n_0}{2}}=\alpha_3^{mn_0}\alpha_4^{m\binom{n_0}{2}}$$

于是 $[\alpha_2^m, \alpha^{n_0}\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}\alpha_4^{n_4}]=\alpha_3^{mn_0}\alpha_4^{m\binom{n_0}{2}}$；

(3°) 由于

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$$[\alpha_1^m, \alpha^{n_0}\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}\alpha_4^{n_4}] = [\alpha_1^m, \alpha^{n_0}\alpha_1^{n_1}] = [\alpha_1^m, \alpha^{n_0}] =$$

$$[\alpha_1, \alpha]^{mn_0}[\alpha_1, \alpha, \alpha]^{m\binom{n_0}{2}}[\alpha_1, \alpha, \alpha, \alpha]^{m\binom{n_0}{3}} = \alpha_2^{mn_0}\alpha_3^{m\binom{n_0}{2}}\alpha_4^{m\binom{n_0}{3}}$$

$$[\alpha^m, \alpha^{n_0}\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}\alpha_4^{n_4}] = [\alpha^m, \alpha^{n_0}\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}] = [\alpha^m, \alpha^{n_0}\alpha_1^{n_1}\alpha_2^{n_2}][\alpha^m, \alpha_3^{n_3}] =$$

$$[\alpha^m, \alpha^{n_0}\alpha_1^{n_1}][\alpha^m, \alpha_2^{n_2}]\alpha_4^{-mn_3} = [\alpha^m, \alpha_1^{n_1}][\alpha^m, \alpha^{n_0}]^{\alpha_1^{n_1}}\alpha_3^{-mn_2}\alpha_4^{-n_2\binom{m}{2}-mn_3} =$$

$$\alpha_2^{-mn_1}\alpha_3^{-n_1\binom{m}{2}-mn_2}\alpha_4^{-n_1\binom{m}{3}-n_2\binom{m}{2}-mn_3}$$

于是

$$[\alpha^{m_0}\alpha_1^{m_1}, \alpha^{n_0}\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}\alpha_4^{n_4}] = [\alpha^{m_0}, \alpha^{n_0}\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}\alpha_4^{n_4}]^{\alpha_1^{m_1}}[\alpha_1^{m_1}, \alpha^{n_0}\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}\alpha_4^{n_4}] =$$

$$\alpha_2^{-m_0n_1+m_1n_0}\alpha_3^{-\binom{m_0}{2}n_1-m_0n_2+m_1\binom{n_0}{2}}\alpha_4^{-\binom{m_0}{3}n_1-\binom{m_0}{2}n_2-m_0n_3+m_1\binom{n_0}{3}}$$

引理 2　设 G 是有限亚交换群，$G=\langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha]=\alpha_{i+1}, \alpha_1^{p^k}=\alpha_2^{p^j}=\alpha_3^p=\alpha_4^p=1, i=1,2,3, k,j \geqslant 1\rangle$，则

（1°）当 $p>3$ 时，$(\alpha^{m_0}\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3}\alpha_4^{m_4})^p = \alpha^{pm_0}\alpha_1^{pm_1}\alpha_2^{pm_2+m_0m_1\binom{p}{2}}$；当 $p=3$ 时，$(\alpha^{m_0}\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3}\alpha_4^{m_4})^p = \alpha^{pm_0}\alpha_1^{pm_1}\alpha_2^{pm_2+m_0m_1\binom{p}{2}}\alpha_3^{m_0^2m_1}\alpha_4^{m_0^2(m_0-1)m_1+m_0^2m_2}$；

（2°）$(\alpha^{m_0}\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3}\alpha_4^{m_4})^{p^2} = \alpha^{p^2m_0}\alpha_1^{p^2m_1}\alpha_2^{p^2m_2+m_0m_1p^2\binom{p}{2}+m_0m_1p\binom{p}{2}}$.

证　（1°）由换位置的性质和引理 1 可得

$$[\alpha^{m_0}, (\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3})^{-1}] = \alpha_2^{m_0m_1}\alpha_3^{\binom{m_0}{2}m_1+m_0m_2}\alpha_4^{\binom{m_0}{3}m_1+\binom{m_0}{2}m_2+m_0m_3} \qquad [\alpha^{m_0}, 2(\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3})^{-1}]=1$$

$$[2\alpha^{m_0}, (\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3})^{-1}] = [\alpha_2^{m_0m_1}\alpha_3^{\binom{m_0}{2}m_1+m_0m_2}\alpha_4^{\binom{m_0}{3}m_1+\binom{m_0}{2}m_2+m_0m_3}, \alpha^{m_0}] = \alpha_3^{m_0^2m_1}\alpha_4^{m_0^2(m_0-1)m_1+m_0^2m_2}$$

$$[3\alpha^{m_0}, (\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3})^{-1}] = [\alpha_3^{m_0^2m_1}\alpha_4^{m_0^2(m_0-1)m_1+m_0^2m_2}, \alpha^{m_0}] = \alpha_4^{m_0^3m_1}$$

由文献[9]第十章的定理 3.3 及 $\alpha_4 \in Z(G)$ 和 $\alpha_4^p=1$ 有

$$(\alpha^{m_0}\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3}\alpha_4^{m_4})^p = (\alpha^{m_0}\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3})^p = \alpha^{m_0p}\prod_{i+j\leqslant p}[i\alpha^{m_0}, j(\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3})^{-1}]^{\binom{p}{i+j}}(\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3})^p$$

当 $p>3$ 时，由文献[10]第一章第四节的例 1 可得 $\alpha_3^p=1$，于是有

$$(\alpha^{m_0}\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3}\alpha_4^{m_4})^p =$$

$$\alpha^{m_0p}[\alpha^{m_0}, (\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3})^{-1}]^{\binom{p}{2}}[2\alpha^{m_0}, (\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3})^{-1}]^{\binom{p}{3}}[3\alpha^{m_0}, (\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3})^{-1}]^{\binom{p}{4}}(\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3})^p =$$

$$\alpha^{pm_0}\alpha_1^{pm_1}\alpha_2^{pm_2+m_0m_1\binom{p}{2}}$$

当 $p=3$ 时，有

$$(\alpha^{m_0}\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3}\alpha_4^{m_4})^p = \alpha^{m_0p}[\alpha^{m_0}, (\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3})^{-1}]^{\binom{3}{2}}[2\alpha^{m_0}, (\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3})^{-1}]^{\binom{3}{3}}(\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3})^p =$$

$$\alpha^{pm_0}\alpha_1^{pm_1}\alpha_2^{pm_2+m_0m_1\binom{p}{2}}\alpha_3^{m_0^2m_1}\alpha_4^{m_0^2(m_0-1)m_1+m_0^2m_2}$$

（2°）由于 $(\alpha^{m_0}\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3}\alpha_4^{m_4})^{p^2} = ((\alpha^{m_0}\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3}\alpha_4^{m_4})^p)^p$，于是当 $p>3$ 时，有

$$(\alpha^{m_0}\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3}\alpha_4^{m_4})^{p^2} = (\alpha^{pm_0}\alpha_1^{pm_1}\alpha_2^{pm_2+m_0m_1\binom{p}{2}})^p = \alpha^{p^2m_0}\alpha_1^{p^2m_1}\alpha_2^{p^2m_2+m_0m_1p^2\binom{p}{2}+m_0m_1p\binom{p}{2}}$$

当 $p=3$ 时，有

$$(\alpha^{m_0}\alpha_1^{m_1}\alpha_2^{m_2}\alpha_3^{m_3}\alpha_4^{m_4})^{p^2} = (\alpha^{pm_0}\alpha_1^{pm_1}\alpha_2^{pm_2+m_0m_1\binom{p}{2}}\alpha_3^{m_0^2m_1}\alpha_4^{m_0^2(m_0-1)m_1+m_0^2m_2})^p = \alpha^{p^2m_0}\alpha_1^{p^2m_1}\alpha_2^{p^2m_2+m_0m_1p^2\binom{p}{2}+m_0m_1p\binom{p}{2}}$$

2　主要结果

定理 1　设 G 是一个 p^6 阶群，则

（1°）如果 $G=\Phi_{25}(321)=\langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha]=\alpha_{i+1}, [\alpha_3, \alpha]=\alpha_4, \alpha^{p^2}=\alpha_4, \alpha_i^{(p)}=\alpha_{i+2}, \alpha_{i+2}^p=1, i=1,2\rangle$，则当 $p>3$ 时，$|\mathrm{Aut}(G)|=p^7$，当 $p=3$ 时，$|\mathrm{Aut}(G)|=p^7(p-1)$；

(2°) 如果 $G=\Phi_{25}(222)=\langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha]=\alpha_{i+1}, [\alpha_3, \alpha]=\alpha_4, \alpha_i^{(p)}=\alpha_{i+2}, \alpha^{p^2}=\alpha_{i+2}^p=1, i=1,2\rangle$，则当 $p>3$ 时，$|\mathrm{Aut}(G)|=p^7(p-1)$，当 $p=3$ 时，$|\mathrm{Aut}(G)|=p^7(p-1)^2$.

证 从(1°),(2°)中 G 的定义关系均可得 $G'=\langle \alpha_2, \alpha_3, \alpha_4\rangle$，$G_3=\langle \alpha_3, \alpha_4\rangle$，$Z(G)=\langle \alpha_4\rangle$，$G''=1$，于是(1°),(2°)中的 G 都是亚交换群. 设 $\sigma\in \mathrm{Aut}(G)$，$\forall x\in G$，以 x' 表示 $\sigma(x)$，则

$$\alpha'=\alpha^{x_{00}}\alpha_1^{x_{01}}\alpha_2^{x_{02}}\alpha_3^{x_{03}}\alpha_4^{x_{04}} \qquad \alpha'_1=\alpha^{x_{10}}\alpha_1^{x_{11}}\alpha_2^{x_{12}}\alpha_3^{x_{13}}\alpha_4^{x_{14}}$$

$$\alpha'_2=\alpha_2^{x_{22}}\alpha_3^{x_{23}}\alpha_4^{x_{24}} \qquad \alpha'_3=\alpha_3^{x_{33}}\alpha_4^{x_{34}} \qquad \alpha'_4=\alpha_4^{x_{44}}$$

其中 $x_{00}x_{11}-x_{10}x_{01}\not\equiv 0$，$0<x_{22}, x_{33}, x_{44}\leqslant p-1$，$0\leqslant x_{ij}\leqslant p-1$，$i=0,1,2,3$，$j=2,3,4$. 由引理 1，我们有

$$[\alpha'_1, \alpha']=[\alpha^{x_{10}}\alpha_1^{x_{11}}\alpha_2^{x_{12}}\alpha_3^{x_{13}}\alpha_4^{x_{14}}, \alpha^{x_{00}}\alpha_1^{x_{01}}\alpha_2^{x_{02}}\alpha_3^{x_{03}}\alpha_4^{x_{04}}]=$$

$$\alpha_2^{x_{00}x_{11}-x_{10}x_{01}}\alpha_3^{-\binom{x_{10}}{2}x_{01}-x_{10}x_{02}+x_{11}\binom{x_{00}}{2}+x_{12}x_{00}}\alpha_4^{-\binom{x_{10}}{3}x_{01}-\binom{x_{10}}{2}x_{02}-x_{10}x_{03}+x_{11}\binom{x_{00}}{3}+x_{12}\binom{x_{00}}{2}+x_{13}x_{00}}$$

这样由 $[\alpha'_1, \alpha']=\alpha'_2$，可得 $x_{00}x_{11}-x_{10}x_{01}\equiv x_{22}$，$-\binom{x_{10}}{2}x_{01}-x_{10}x_{02}+x_{11}\binom{x_{00}}{2}+x_{12}x_{00}\equiv x_{23}$，$-\binom{x_{10}}{3}x_{01}-\binom{x_{10}}{2}x_{02}-x_{10}x_{03}+x_{11}\binom{x_{00}}{3}+x_{12}\binom{x_{00}}{2}+x_{13}x_{00}\equiv x_{24}$. 同理，由 $[\alpha'_2, \alpha']=\alpha'_3$，$[\alpha'_3, \alpha']=\alpha'_4$，有 $x_{22}x_{00}\equiv x_{33}$，$x_{22}\binom{x_{00}}{2}+x_{23}x_{00}\equiv x_{34}$，$x_{33}x_{00}\equiv x_{44}$.

(1°) 当 $p>3$ 时，由定义关系可知 $\alpha_1^{p^2}=\alpha_2^{p^2}=\alpha_3^p=\alpha_4^p=1$. 于是由 $(\alpha')^{p^2}=\alpha'_4$ 有 $(\alpha^{x_{00}}\alpha_1^{x_{01}}\alpha_2^{x_{02}}\alpha_3^{x_{03}}\alpha_4^{x_{04}})^{p^2}=\alpha_4^{x_{44}}$，由引理 2 得 $\alpha_4^{x_{00}}=\alpha_4^{x_{44}}$，这样 $x_{00}\equiv x_{44}$. 同理，由 $\alpha'^{(p)}_i=\alpha'_{i+2}$，$\alpha'^{p}_{i+2}=1$ $(i=1,2)$ 可以得到 $x_{22}\equiv x_{44}$，$x_{10}\equiv 0$. 于是，根据同余式的运算可得 $x_{00}\equiv x_{11}\equiv x_{22}\equiv x_{33}\equiv x_{44}\equiv 1$，$x_{12}\equiv x_{23}\equiv x_{34}$，$x_{13}\equiv x_{24}$. 则当 $p>3$ 时，$|\mathrm{Aut}(G)|=p^7$.

类似地，当 $p=3$ 时，由定义关系可知 $\alpha_i^3=1$ $(i=1,2,3,4)$. 通过讨论可得 $x_{00}\equiv x_{44}$，$x_{10}\equiv 0$，于是有 $x_{33}\equiv 1$，$x_{11}\equiv x_{00}^{-2}$，$x_{22}\equiv x_{00}^{-1}$，$x_{12}\equiv x_{00}^{-1}(x_{23}-x_{00}^{-2}\binom{x_{00}}{2})$，$x_{24}\equiv x_{00}^{-2}\binom{x_{00}}{3}+x_{00}^{-1}(x_{23}-x_{00}^{-2}\binom{x_{00}}{2})\binom{x_{00}}{2}+x_{13}x_{00}$，$x_{34}\equiv x_{00}^{-1}\binom{x_{00}}{2}+x_{23}x_{00}$. 则当 $p=3$ 时，有 $|\mathrm{Aut}(G)|=p^7(p-1)$.

(2°) 类似(1°)，当 $p>3$ 时，还有 $x_{22}\equiv x_{44}$，$x_{11}\equiv x_{33}$，$\frac{p-1}{2}(x_{22}-x_{11})+x_{12}\equiv x_{34}$，$x_{10}\equiv 0$，于是可得 $x_{00}\equiv 1$，$x_{10}\equiv 0$，$x_{11}\equiv x_{22}\equiv x_{33}\equiv x_{44}$，$x_{12}\equiv x_{23}\equiv x_{34}$，$x_{13}\equiv x_{24}$. 则当 $p>3$ 时有 $|\mathrm{Aut}(G)|=p^7(p-1)$.

当 $p=3$ 时，有 $x_{10}\equiv 0$，$x_{00}\equiv x_{11}^{-1}x_{22}$，$x_{33}\equiv x_{11}^{-1}x_{22}^2$，$x_{44}\equiv x_{11}^{-2}x_{22}^3$，$x_{23}\equiv x_{11}\binom{x_{11}^{-1}x_{22}}{2}+x_{12}x_{11}^{-1}x_{22}$，$x_{24}\equiv x_{11}\binom{x_{11}^{-1}x_{22}}{3}+x_{12}\binom{x_{11}^{-1}x_{22}}{2}+x_{13}x_{11}^{-1}x_{22}$，$x_{34}\equiv x_{22}\binom{x_{11}^{-1}x_{22}}{2}+x_{22}\binom{x_{11}^{-1}x_{22}}{2}+x_{12}x_{11}^{-2}x_{22}^2$. 则当 $p=3$ 时，$|\mathrm{Aut}(G)|=p^7(p-1)^2$.

定理 2 设 G 是一个 p^6 阶群，则

(1°) 如果 $G=\Phi_{26}(321)=\langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha]=\alpha_{i+1}, [\alpha_3, \alpha]=\alpha_4, \alpha^{p^2}=\alpha_4^{\nu}, \alpha_i^{(p)}=\alpha_{i+2}^{\nu}, \alpha_{i+2}^p=1, i=1,2\rangle$，则当 $p>3$ 时，$|\mathrm{Aut}(G)|=p^7$，当 $p=3$ 时，$|\mathrm{Aut}(G)|=2p^6$；

(2°) 如果 $G=\Phi_{26}(222)=\langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha]=\alpha_{i+1}, [\alpha_3, \alpha]=\alpha_4, \alpha_i^{(p)}=\alpha_{i+2}^{\nu}, \alpha^{p^2}=\alpha_{i+2}^p=1, i=1,2\rangle$，则当 $p>3$ 时，$|\mathrm{Aut}(G)|=p^7(p-1)$，当 $p=3$ 时，$|\mathrm{Aut}(G)|=2p^6(p-1)$.

证 类似于定理 1 的讨论，可以得到

$$x_{00}x_{11}-x_{10}x_{01}\equiv x_{22} \qquad -\binom{x_{10}}{2}x_{01}-x_{10}x_{02}+x_{11}\binom{x_{00}}{2}+x_{12}x_{00}\equiv x_{23}$$

$$-\binom{x_{10}}{3}x_{01}-\binom{x_{10}}{2}x_{02}-x_{10}x_{03}+x_{11}\binom{x_{00}}{3}+x_{12}\binom{x_{00}}{2}+x_{13}x_{00}\equiv x_{24}$$

$$x_{22}x_{00} \equiv x_{33} \qquad x_{22}\binom{x_{00}}{2} + x_{23}x_{00} \equiv x_{34} \qquad x_{33}x_{00} \equiv x_{44}$$

（1°）当 $p > 3$ 时，由定义关系可知 $\alpha_1^{p^2} = \alpha_2^{p^2} = \alpha_3^p = \alpha_4^p = 1$. 于是由引理 2 以及 $(\alpha')^{p^2} = \alpha'^{\nu}_4$，有 $\alpha_4^{x_{00}} = \alpha_4^{\nu x_{44}}$，而 $(\nu, p) = 1$，则 $x_{00} \equiv x_{44}$；由 $(\alpha'_2)^{(3)} = \alpha'^{\nu}_4$，有 $x_{22} \equiv x_{44}$；由 $(\alpha'_1)^{p^2} = 1$，有 $x_{10} \equiv 0$；由 $(\alpha'_1)^{(p)} = \alpha'^{\nu}_3$，有 $x_{11} \equiv x_{33}$，$\frac{p-1}{2}(x_{22} - x_{11}) + x_{12} \equiv x_{34}$. 则 $x_{00} \equiv x_{11} \equiv x_{22} \equiv x_{33} \equiv x_{44} \equiv 1$，$x_{12} \equiv x_{23} \equiv x_{34}$，$x_{13} \equiv x_{24}$，故当 $p > 3$ 时，$|\mathrm{Aut}(G)| = p^7$.

当 $p = 3$ 时，由 $(\alpha')^{3^2} = \alpha'^{\nu}_4$，有 $x_{00} \equiv x_{44}$；由 $(\alpha'_1)^{3^2} = 1$，有 $\alpha_4^{\nu x_{10}} = 1$，即 $x_{10} \equiv 0$；由 $(\alpha'_2)^{(3)} = \alpha'^{\nu}_4$，有 $(\nu - 1)(x_{22} - x_{44}) \equiv 0$，由于 $p = 3$，则 $\nu \equiv -1 \pmod 3$，当然 $\nu - 1 \not\equiv 0 \pmod 3$，则 $x_{22} \equiv x_{44}$；由 $(\alpha'_1)^{(p)} = \alpha'^{\nu}_3$，有 $x_{11} \equiv x_{33}$，$2x_{11} + x_{12} \equiv 2x_{34} + 2x_{44}$. 这样可以得 $x_{11} \equiv x_{33} \equiv 1$，$x_{00}^2 \equiv 1$，于是 $x_{00} \equiv 1$ 或 $x_{00} \equiv 2$. 如果 $x_{00} \equiv 1$，有 $x_{11} \equiv x_{22} \equiv x_{33} \equiv x_{44} \equiv 1$，$x_{12} \equiv x_{23} \equiv x_{34} \equiv 0$，$x_{24} \equiv x_{13}$；如果 $x_{00} \equiv 2$，有 $x_{00} \equiv x_{12} \equiv x_{22} \equiv x_{23} \equiv x_{44} \equiv 2$，$x_{34} \equiv 0$，$x_{24} \equiv 2 + 2x_{13}$，故当 $p = 3$ 时，$|\mathrm{Aut}(G)| = 2p^6$.

（2°）类似（1°），当 $p > 3$ 时，有 $x_{22} \equiv x_{44}$，$x_{11} \equiv x_{33}$，$\frac{p-1}{2}(x_{22} - x_{11}) + x_{12} \equiv x_{34}$，$x_{10} \equiv 0$，于是当 $p > 3$ 时，即有 $|\mathrm{Aut}(G)| = p^7(p-1)$.

当 $p = 3$ 时，有 $x_{22} \equiv x_{44}$，$x_{10} \equiv 0$，$x_{11} \equiv x_{33}$，$2x_{11} + x_{12} \equiv 2x_{34} + 2x_{44}$，这样可以得 $x_{00}^2 \equiv 1$，于是 $x_{00} \equiv 1$ 或 $x_{00} \equiv 2$. 如果 $x_{00} \equiv 1$，有 $x_{11} \equiv x_{22} \equiv x_{33} \equiv x_{44}$，$x_{12} \equiv x_{23} \equiv x_{34} \equiv 0$，$x_{24} \equiv x_{13}$；如果 $x_{00} \equiv 2$，有 $x_{11} \equiv x_{33}$，$x_{22} \equiv x_{44} \equiv 2x_{11}$，$x_{34} \equiv 0$，$x_{12} \equiv x_{23} \equiv 2x_{11}$，$x_{24} \equiv 2x_{11} + 2x_{13}$，故当 $p = 3$ 时，$|\mathrm{Aut}(G)| = 2p^6(p-1)$.

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The Orders of the Automorphism Groups of Two Families of the Groups of Order p^6

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Abstract：According the classification of the groups of order p^6，utilizing the defining relation of the generators of the meta-Abel group，some properties of automorphism and the operations of congruence，the orders of automorphism groups of Φ_{25} and Φ_{26}，which are two families of groups of order p^6，are obtained.

Key words: meta-Abel group; defining relation; automorphism group; order

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