η-Proximal Point Algorithm for a System of Generalized Implicit Quasi-Variational-Like Inclusion Problems

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Abstract: In this paper, we consider a new system of generalized implicit quasi-variational-like inclusion problems (GIIQP) involving \(\eta\)-subdifferentiable mapping and \(\eta\)-proximal mapping in real Hilbert spaces. By applying some lemmas, we develop a new \(\eta\)-proximal point algorithm of Mann and Ishikawa type for finding the approximate solutions of the generalized implicit quasi-variational-like inclusion problems. The convergence criteria of the sequences of the approximate solutions generated by the new algorithm are also discussed.

Key words: generalized implicit quasi-variational-like inclusion problems; \(\eta\)-proximal point algorithm; Hilbert space

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Variational inclusion problems are the most interesting and intensively studied of classics mathematical problems. For the past years, many existence results and iterative algorithms for various variational inequality and variational inclusion problems have been studied, see [1–7].

Recently, many systems of variational inequality problems were introduced and studied. [3–5] first considered a system of scalar variational inequalities, [6,7] introduced a system of quasi-variational inequality problems and proved its existence theorem by maximal element theorems.

Throughout the paper unless otherwise stated, let \(I = \{1, 2\}\) be an index set and for each \(i \in I\), let \(H_i\) be a real Hilbert space whose inner product and norm are denoted by \(\langle \cdot, \cdot \rangle\) and \(\| \cdot \|\), respectively.

[2] studied the following problems. Let \(F_i : H_i \times H_i \rightarrow H_i, \eta_i : H_i \times H_i \rightarrow H_i, b_i : H_i \times H_i \rightarrow \mathbb{R}\), where \(i \in \{1, 2\}\), find \((x, y) \in H_1 \times H_2\) such that

\[
\langle F_1(x, y), \eta_1(v_1, x) \rangle + b_1(x, v_1) - b_1(x, x) \geq 0 \quad \forall v_1 \in H_1
\]

\[
\langle F_2(x, y), \eta_2(v_2, y) \rangle + b_2(y, v_2) - b_2(y, y) \geq 0 \quad \forall v_2 \in H_2
\]

which is called a system of generalized variational-like inequality problems.

The purpose of this paper is to introduce and study a new system of generalized implicit quasi-variational-like inclusion problems involving \(\eta\)-subdifferentiable mapping and \(\eta\)-proximal mapping in real Hilbert space. By applying \(\eta\)-proximal point technique of Mann and Ishikawa type, we obtained an existence theorem of solution of the GIIQP. Our algorithm and results improve and generalize many known re-
suits in recent literatures.

By the definition of \( \eta^- \)-subdifferential\(^{[1]} \) of \( \phi_1 : H_1 \longrightarrow R \cup \{+\infty\} \) and \( J^\phi_1 (x) \), we have \( x - u \in \rho \Delta \phi_1 (u) \). It follows that

\[
J^\phi_1 (x) = (I + \rho \Delta \phi_1)^{-1} (x)
\]

where \( I \) is the identity mapping on \( H_1 \).

Now, for each \( i \in I \), let \( T_i : H_1 \times H_2 \longrightarrow H_1, g_i : H_1 \longrightarrow H_1, \) and \( \eta_i : H_1 \times H_1 \longrightarrow \) \( R \cup \{+\infty\} \) be a proper functional such that for each fixed \( y \in H_1 \), \( \phi_i (\cdot, y) : H_1 \longrightarrow R \) is lower semicontinuous and \( \eta_i^- \)-subdifferentiable on \( H_1 \) and \( g_i (H_1) \cap \text{dom} \Delta \phi_i (\cdot, y) \neq \emptyset \). We will consider the following system of generalized implicit quasi-variational-like inclusion problems (GIQV-LIP).

Find \( (x, y) \in H_1 \times H_2 \) such that

\[
\langle T_1 (x, y), \eta (v_1, g_1 (x)) \rangle_1 \geq \phi_1 (g_1 (x), x) - \phi_1 (v_1, x) \quad \forall v_1 \in H_1 \quad (1)
\]

\[
\langle T_2 (x, y), \eta (v_2, g_2 (y)) \rangle_2 \geq \phi_2 (g_2 (y), y) - \phi_2 (v_2, y) \quad \forall v_2 \in H_2 \quad (2)
\]

**Remark 1** If \( H = H_1 = H_2, T_1 = T_2, \phi = \phi_1, \eta = \eta_1, g = g_1 = g_2, T_1 (x, y) = T (x) - A (y) \), then the GIQVLIP \((1),(2)\) reduce to the following general quasi-variational-like inclusion problem (GQV-LIP):

Find \( x \in H \) such that \( g (x) \in \text{dom} \Delta \phi (\cdot, x) \) and

\[
\langle T (x) - A (x), \eta (v, g (x)) \rangle \geq \phi (g (x), x) - \phi (v, x) \quad \forall v \in H \quad (3)
\]

Problem \((3)\) was introduced and studied by \([1]\), which includes general quasi-variational-like inclusions.

We first transfer the GIQVLIP \((1),(2)\) into a fixed-point problem.

**Theorem 1** \((x^*, y^*) \in H_1 \times H_2 \) is a solution of the GIQVLIP \((1),(2)\) if and only if \((x^*, y^*)\) satisfies the following relations:

\[
g_1 (x) = J^{\phi_1 (\cdot, x)} (g_1 (x) - \rho T_1 (x, y)) \quad (4)
\]

\[
g_2 (y) = J^{\phi_2 (\cdot, y)} (g_2 (y) - \rho T_2 (x, y)) \quad (5)
\]

where \( J^{\phi_1 (\cdot, x)} = (I + \rho \Delta \phi_1 (\cdot, x))^{-1} \) is the \( \eta^- \)-proximal mapping of \( \phi_1 (\cdot, x) \), \( I \) is the identity mapping on \( H_1 \) and \( \rho > 0 \) is a constant.

**Proof** Assume that \((x^*, y^*) \in H_1 \times H_2\) satisfies relations \((4),(5)\), i.e.

\[
g_1 (x^*) = J^{\phi_1 (\cdot, x^*)} (g_1 (x^*) - \rho T_1 (x^*, y^*))
\]

\[
g_2 (y^*) = J^{\phi_2 (\cdot, y^*)} (g_2 (y^*) - \rho T_2 (x^*, y^*))
\]

The equalities hold if and only if

\[
-T_1 (x^*, y^*) \in \Delta \phi_1 (g_1 (x^*), x^*)
\]

\[
-T_2 (x^*, y^*) \in \Delta \phi_2 (g_2 (y^*), y^*)
\]

By the definition of \( \eta_i^- \)-subdifferential of \( \phi_i (\cdot, y) \) \( (\phi_i (\cdot, y^*) \) \), the above relations hold if and only if

\[
\phi_1 (v_1, x^*) - \phi_1 (g_1 (x^*), x^*) \geq \langle - T_1 (x^*, y^*), \eta (v_1, g_1 (x^*)) \rangle_1 \quad \forall v_1 \in H_1
\]

\[
\phi_2 (v_2, y^*) - \phi_2 (g_2 (y^*), y^*) \geq \langle - T_2 (x^*, y^*), \eta (v_2, g_2 (y^*)) \rangle_2 \quad \forall v_2 \in H_2
\]

and hence

\[
\langle T_1 (x^*, y^*), \eta (v_1, g_1 (x^*)) \rangle_1 \geq \phi_1 (g_1 (x^*), x^*) - \phi_1 (v_1, x^*) \quad \forall v_1 \in H_1
\]

\[
\langle T_2 (x^*, y^*), \eta (v_2, g_2 (y^*)) \rangle_2 \geq \phi_2 (g_2 (y^*), y^*) - \phi_2 (v_2, y^*) \quad \forall v_2 \in H_2
\]

i.e. \((x^*, y^*)\) is a solution of the GIQVLIP \((1),(2)\).

**Remark 2** Equations \((4),(5)\) can be written as

\[
x = x - g_1 (x) + J^{\phi_1 (\cdot, x)} (g_1 (x) - \rho T_1 (x, y))
\]

\[
y = y - g_2 (y) + J^{\phi_2 (\cdot, y)} (g_2 (y) - \rho T_2 (x, y))
\]

These fixed-point formulations enable us to suggest the following iterative algorithm.
Algorithm 1  For each \( i \in I \), let \( T_i : H_1 \times H_2 \longrightarrow H_1 \) \( , \ g_i : H_1 \longrightarrow H_1 \) \( , \) and \( \eta_i : H_1 \times H_2 \longrightarrow \mathbb{R} \) be single-valued mappings and \( \phi_i : H_1 \times H_2 \longrightarrow \mathbb{R} \ union \ (\pm \infty) \) be a proper functional such that for each fixed \( y \in H_1 \), \( \phi_i (\cdot , y) : H_1 \longrightarrow \mathbb{R} \) is lower semicontinuous and \( \eta_i \) -subdifferentiable on \( H_1 \) and \( g_i (H_1) \bigcap \text{dom} \phi_i (\cdot , y) \neq \emptyset \). For any given \( (x_n , y_n) \in H_1 \times H_2 \), the iterative sequences \( x_n \), \( y_n \) are defined by

\[
x_{n+1} = x_n - g_i (x_n) + J_{\phi_i (x_n , y_n)}^\tau (x_n , y_n) - \rho T_i (x_n , y_n) \quad n = 0, 1, 2, \ldots
\]

\[
y_{n+1} = y_n - g_i (y_n) + J_{\phi_i (y_n , x_n)}^\tau (y_n , x_n) - \rho T_i (x_n , y_n) \quad n = 0, 1, 2, \ldots
\]

where \( \rho > 0 \) is a constant.

Remark 3  The Algorithm 1 includes many known algorithms in the literature as special cases. Now, we establish the existence of the unique solution of the GIQVLIP (1), (2), and the convergence of iterative sequences generalized by the Algorithm 1.

Theorem 2  Let the mapping \( T_1 : H_1 \times H_2 \longrightarrow H_1 \) be \( a_1 \) -strongly monotone\(^{[2]} \) in the first argument and \( (\beta_1 , \xi_1) \) -Lipschitz continuous\(^{[2]} \) and the mapping \( T_2 : H_1 \times H_2 \longrightarrow H_2 \) be \( a_2 \) -strongly monotone in the second argument and \( (\beta_2 , \xi_2) \) -Lipschitz continuous. For each \( i \in I \), let the mapping \( \eta_i : H_1 \times H_2 \longrightarrow \mathbb{R} \) be \( \delta_i \) -strongly monotone and \( \tau \) -Lipschitz continuous such that \( \eta_i (x , y) = - \eta_i (y , x) \), \( \forall x , y \in H_1 \) and for each given \( x \in H_1 \), the function \( h_i (y , u) = \langle x - u , \eta_i (y , u) \rangle \) is 0-DQC in \( y \). Let \( \phi_i : H_1 \times H_2 \longrightarrow \mathbb{R} \ union \ (\pm \infty) \) be such that for each fixed \( y \in H_1 \), \( \phi_i (\cdot , y) : H_1 \longrightarrow \mathbb{R} \) is lower semicontinuous and \( \eta_i \) -subdifferentiable proper functional satisfying \( g_i (H_1) \bigcap \text{dom} \phi_i (\cdot , y) \neq \emptyset \). Suppose that there exists a constant \( \rho > 0 \) such that for each \( x , y , z \in H_1 \)

\[
\| J_{\phi_i (x , y)}^\tau (z) - J_{\phi_i (x , y)}^\tau (z) \| \leq \mu_i \| x - y \|,
\]

and

\[
k_i = \mu_i + (1 + \frac{\tau_i}{\delta_i}) \sqrt{1 - 2\lambda_i + \sigma_i^2}
\]

\[
a_1 \tau_i^2 \delta_i^2 > \tau_i \delta_i \beta_i (1 - k_1) \delta_i^2 + \sqrt{\beta_i \tau_i^2 \delta_i^2 - \beta_i \tau_i^2 \delta_i^2 (1 - k_1)^2} - \tau_i \delta_i^2
\]

\[
a_2 \tau_i^2 \delta_i^2 > \tau_i \delta_i \gamma_i (1 - k_2) \delta_i^2 + \sqrt{\gamma_i \tau_i^2 \delta_i^2 - \gamma_i \tau_i^2 \delta_i^2 (1 - k_2)^2} - \tau_i \delta_i^2
\]

\[
| \rho - \frac{a_1}{b_1} | < \frac{\sqrt{b_i (\delta_i \delta_i (1 - k_1)^2 - \tau_i \delta_i^2) + a_i}}{b_i}
\]

\[
| \rho - \frac{a_2}{b_2} | < \frac{\sqrt{b_i (\delta_i \delta_i (1 - k_2)^2 - \tau_i \delta_i^2) + b_i}}{b_i}
\]

where

\[
a_1 = a_1 \tau_i^2 \delta_i^2 - \tau_i \delta_i \beta_i (1 - k_1) \delta_i^2 \quad a_2 = a_2 \tau_i^2 \delta_i^2 - \tau_i \delta_i \gamma_i (1 - k_2) \delta_i^2
\]

\[
b_1 = \beta_i \tau_i^2 \delta_i^2 - \beta_i \tau_i^2 \delta_i^2 \quad b_2 = \gamma_i \tau_i^2 \delta_i^2 - \gamma_i \tau_i^2 \delta_i^2
\]

Then the GIQVLIP (1), (2) have a unique solution \( (x^* , y^*) \in H_1 \times H_2 \).

Proof  By Theorem 1, it is sufficient to show that there exist a unique \( (x^* , y^*) \in H_1 \times H_2 \) satisfying Equations (4) and (5).

By Algorithm 1 and Theorem 2.8 of [1]

\[
\| x_{n+1} - x_n \| \leq \| x_n - g_i (x_n) + J_{\phi_i (x_n , y_n)}^\tau (x_n , y_n) - g_i (x_{n-1}) \| - \| J_{\phi_i (x_{n-1} , y_{n-1})}^\tau (x_{n-1} , y_{n-1}) - g_i (x_{n-1}) \| \| x_n - x_{n-1} \|
\]

\[
\leq \| x_n - x_{n-1} - (g_i (x_{n-1}) - g_i (x_{n-1})) \| + \| J_{\phi_i (x_{n-1} , y_{n-1})}^\tau (x_{n-1} , y_{n-1}) - J_{\phi_i (x_{n-1} , y_{n-1})}^\tau (x_{n-1} , y_{n-1}) \| + \| J_{\phi_i (x_{n-1} , y_{n-1})}^\tau (x_{n-1} , y_{n-1}) - J_{\phi_i (x_{n-1} , y_{n-1})}^\tau (x_{n-1} , y_{n-1}) \| \| x_n - x_{n-1} \|
\]

By Theorem 2.10 of [1], we know the \( \eta_i \) -proximal mapping \( J_{\phi_i}^\tau \) of \( \phi_i \) is \( \tau_i / \delta_i \) -Lipschitz continuous, then we have
\[ \| x_{n+1} - x_n \|_1 \leq \| x_n - x_{n-1} - (g_1(x_n) - g_1(x_{n-1})) \|_1 + \frac{\tau_1}{\delta_1} \| g_1(x_n) - \rho T_1(x_n, y_n) - (g_1(x_{n-1}) - \rho T_1(x_{n-1}, y_{n-1})) \|_1 + \mu_1 \| x_n - x_{n-1} \|_1 \]  

(6)

Since \( g_1 \) is \( \sigma_1 \)-Lipschitz continuous, \( \lambda_1 \)-strongly monotone
\[ \| x_n - x_{n-1} - (g_1(x_n) - g_1(x_{n-1})) \|_1 \leq \sqrt{1 - 2\lambda_1 + \sigma_1^2} \| x_n - x_{n-1} \|_1 \]  

(7)

Since \( T_1 \) is \( (\beta_1, \xi_1) \)-Lipschitz continuous, \( \alpha_1 \)-strongly monotone in the first argument
\[ \| g_1(x_n) - \rho T_1(x_n, y_n) - (g_1(x_{n-1}) - \rho T_1(x_{n-1}, y_{n-1})) \|_1 + \| x_n - x_{n-1} - \rho (T_1(x_n, y_n) - T_1(x_{n-1}, y_{n-1})) \|_1 + \rho \| T_1(x_n, y_n) - T_1(x_{n-1}, y_{n-1}) \|_1 \leq \sqrt{1 - 2\lambda_1 + \sigma_1^2} + \sqrt{1 - 2\lambda_1 + \sigma_1^2} \| x_n - x_{n-1} \|_1 + \rho \| y_n - y_{n-1} \|_2 \]  

(8)

From the above inequalities (6)–(8), we obtain
\[ \| x_{n+1} - x_n \|_1 \leq \left( \left(1 + \frac{\tau_1}{\delta_1} \right) \sqrt{1 - 2\lambda_1 + \sigma_1^2} + \frac{\tau_1}{\delta_1} \sqrt{1 - 2\alpha_1 + \beta_1^2} + \mu_1 \right) \| x_n - x_{n-1} \|_1 + \frac{\varphi_1 \xi_1}{\delta_1} \| y_n - y_{n-1} \|_2 \]  

(9)

Similarly, we have
\[ \| y_{n+1} - y_n \|_2 \leq \left( \left(1 + \frac{\tau_2}{\delta_2} \right) \sqrt{1 - 2\lambda_2 + \sigma_2^2} + \frac{\tau_2}{\delta_2} \sqrt{1 - 2\alpha_2 + \xi_2^2} + \mu_2 \right) \| y_n - y_{n-1} \|_2 + \frac{\varphi_2 \xi_2}{\delta_2} \| x_n - x_{n-1} \|_1 \]  

(10)

Let
\[ \theta_1 = \left(1 + \frac{\tau_1}{\delta_1} \right) \sqrt{1 - 2\lambda_1 + \sigma_1^2} + \frac{\tau_1}{\delta_1} \sqrt{1 - 2\alpha_1 + \beta_1^2} + \mu_1 + \frac{\varphi_1 \xi_1}{\delta_1} \]  
\[ \theta_2 = \left(1 + \frac{\tau_2}{\delta_2} \right) \sqrt{1 - 2\lambda_2 + \sigma_2^2} + \frac{\tau_2}{\delta_2} \sqrt{1 - 2\alpha_2 + \xi_2^2} + \mu_2 + \frac{\varphi_2 \xi_2}{\delta_2} \]

From (9), (10), we have
\[ \| x_{n+1} - x_n \|_1 + \| y_{n+1} - y_n \|_2 \leq \min(\theta_1, \theta_2) \| x_n - x_{n-1} \|_1 + \| y_n - y_{n-1} \|_2 \]  

(11)

Now, we define that the norm \( \| \cdot \| \) on \( H_1 \times H_2 \) by
\[ \| (u, v) \|_\infty = \| u \|_1 + \| v \|_2 \quad \forall (u, v) \in H_1 \times H_2 \]

It observe that \( (H_1 \times H_2, \| \cdot \|) \) is a Banach space. Hence, (11) implies that
\[ \| (x_{n+1}, y_{n+1}) - (x_n, y_n) \| \leq \min(\theta_1, \theta_2) \| (x_n, y_n) - (x_{n-1}, y_{n-1}) \| \]  

(12)

By the conditions, it follows that \( \theta_1, \theta_2 \in (0, 1) \) and hence (12) implies that \( (x_n, y_n) \) is a Cauchy sequence in \( H_1 \times H_2 \) as \( n \to \infty \). From the fact \( T_1 \), \( \eta \), \( \phi \), are continuous. We have
\[ g_1(x^*) = J_{\rho}^{\psi(x^*)} (g_1(x^*) - \rho T_1(x^*, y^*)) \]
\[ g_2(y^*) = J_{\rho}^{\psi(y^*)} (g_2(y^*) - \rho T_2(x^*, y^*)) \]

Therefore \( (x^*, y^*) \) is a solution of GIQVILP \((1), (2)\). This completes the proof.
Remark 4

(i) Using the method presented in this paper, one can extend GIQVLIP (1), (2) to the system of $n$-generalized implicit quasi-variational-like inclusion problems.

(ii) It is of further research interest to extend the method presented in this paper for iterative approximations of solution of GIQVLIP involving set-valued mappings.

References:


一组广义隐拟似变分包含问题的 $\eta$-逼近点算法

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摘要：在实 Hilbert 空间中讨论了一组新的关于 $\eta$-次可微算子和 $\eta$-逼近算子的广义隐拟似变分包含问题，提出了一个逼近其解的新的 $\eta$-逼近点算法，还讨论了由算法得到的序列的逼近特征。

关键词：广义隐拟似变分包含问题；$\eta$-逼近点算法；Hilbert 空间